Math Reference U

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• Ctrl + End	Go to the last page
• Ctrl + Shift	Highlight words
• Ctrl + Click	Highlight a line

Basics

Symbols

Symbol	Term	Negation
+	Addition operator: plus, add, sum	−
−	Subtraction operator: minus, subtract, difference	+
×	Multiplication operator: multiply, product, sum	÷
·	Bullet operator: multiply; notations: variable or constant to bracket or variable	÷
÷	Division operator: divide, quotient	×
/	Slash operator: divide; notations: fractional	×
=	Equals: Total of equation	≠
≈	Almost equal or approximately	≉
<	Less than: requires 2 constants or variables	≮
≤	Less than or equal: requires 2 constants or variables	≰
>	Greater than: requires 2 constants or variables	≯
≥	Greater than or equal: requires 2 constants or variables	≱
± ∓	Plus minus: positive to negative; Minus Plus: negative to positive	∓ ±
∞	Infinity	−∞
°	Degree (360)	
Δ	Increment; Change in; Delta; or triangle	
∇	Decline	
π	Pi constant: 3.1415926535898	$-\pi$
ϕ	Phi constant/golden ratio: 1.61803399	$-\phi$
√	Square root operator; negation to square	y^x ^
y^x ^	Exponent operator: to the power of, multiply; Exponent operator	√
y_x _	Subscript: Used to array variables	
%	Percentage: expressed as a fraction when over 100 or decimal when less than 1	
!	Factorial: multiplies all terms from an integer down to 1, can't be less than 1	
□	Isolated term	
( ) [ ]	Brackets: alternate between square and curved; also interval notation	
∟	Right Angle: 90 degrees	
∠	Angle	
∡	Measured Angle	
∢	Spherical angle	
⊾	Right angle with arc	
⊿	Right triangle	
⋕	Equal and parallel to	
⊥	Perpendicular to	
∤	Does not divide	
∥	Parallel	∦
∶	Ratio: comparing 2 or more values	
∈	Element of: relations	
ℝ	Real Number	
∵	Because; since	
∴	Therefore	
∪	Union: or inclusive	
θ	Theta: objective angle to find	
α	Alpha: variable notation	
∎	End	

Equations

Name	Equation
Mixed Number	$z\frac{x}{y} = (yz + x)$
Fraction Exponent	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
Multiplying Exponents	$a^n \times a^m = a^{n+m}$
Dividing Exponents	$a^n \times a^m = a^{n-m}$
Bracket Exponent	$\left(a^b\right)^c = a^{bc}$
Distributive Property Exponent	$(ab)^c = (a^1b^1)^c = (a^{1c}b^{1c})$
Distributive Property	$a(x + y) = ax + ay$
Length Line Segment	$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint Line Segment	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Line Substitution	$y - y_1 = m(x - x_1)$
Circle Formula	$x^2 + y^2 = r^2$
Circle Centroid	$(x - p)^2 + (y - q)^2 = r^2$
Sum of Interior Angles	$180(n - 2)$
Factoring Trinomial	$ax^2 + bx + c$
Quadratic Function	$y = ax^2 + k$
Expanded Quadratic Function	$y = a(x - h)^2 + k$
Square Quadratic Function	$y = ax^2 + bx + c$
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Trigonometry Functions	SOH-CAH-TOA
Sine Law	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Reverse Sine Law	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Law	$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$ $c^2 = a^2 + b^2 - 2ab\cos C$
Reverse Cosine Law	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Arithmetic Sequence	$t_n = a + (n-1)d$
Arithmetic Series	$S_n = \frac{n}{2}(a + t_n)$
Alternative Arithmetic Series	$S_n = \frac{n}{2}(2a + (n-1)d)$
Geometric Sequence	$t_n = ar^{n-1}$
Geometric Series	$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$
Compound Interest	$A = P(1+i)^n$
Present Value	$P = A(1+i)^{-n}$
Ordinary Annuity	$A = \frac{R((1+i)^n - 1)}{i}$
Present Ordinary Annuity	$P = \frac{R(1-(1+i)^{-n})}{i}$
Power Function	$y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 + a_0$
First Difference	$c = a(n!)$
Polynomial Families	$y = k(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 - ab + b^2)$
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Physics

Name	Equation
Density	$D = \frac{m}{V}$
Motion	$d = v - t$
Average Velocity and Acceleration	$a = \frac{v_f - v_i}{t}$
Uniform Motion with Constant Acceleration	$d = v_i \cdot t + \frac{1}{2} \cdot a \cdot t^2$
Newton's Second Law	$F = m - a$
Gravity	$F_g = \frac{G \cdot m_1 \cdot m_2}{d^2}$
Momentum	$p = m - v$
Work and Power	$W = F - dP = \frac{W}{t}$
Energy	$K.E. = \frac{1}{2} \cdot m \cdot v^2$
Static Electricity	$F_E = \frac{k \cdot q_1 \cdot q_2}{d^2}$
Current Electricity	$V = \frac{W}{q} l = \frac{q}{t}$ $W = V \cdot I \cdot t$ $P = V \cdot I$
Energy Transfer	$q = m \cdot c\Delta T$

General

Terms

- **Expression**: a mathematical sentence without an equal sign ($=$). The only way to solve an **expression** is through **substitution**
- **Equation**: a mathematical sentence with an equal sign ($=$)
- An **equation** is a math statement that states 2 **expressions** are equal

Example: $-3x + 3 = 2x - 2$

- A **solution** is the value of the **variable** that makes an **equation**

Example:
$$-3x + 3 = 2x - 2$$
$$-3x - 2x = -2 - 3$$
$$\frac{-5x}{-5} = -\frac{-5}{-5}$$
$$x = 1$$

- A formula describes an **algebraic** relationship between 2 or more **variables**
- Q.E.D. means that what you have set out to prove has been proven true

Global Variables

- A: Area
- P: Perimeter
- V: Volume
- l: Length
- w: Width
- h: Height
- b: Base
- m: Slope
- v: Velocity
- d: Distance
- t: Time
- I: Interest
- i: Imaginary number
- p: Principle
- r: Rate or hypotenuse
- x: Horizontal axis
- y: Vertical axis

Adding

Adding in sequence in linear

Example: $25 + 37 = 62$

Adding with multiple values in linear

Example: $25 + 25 + 89 + 45$

- When **adding** with **decimals**, align decimals up then solve

Subtracting

Subtracting in sequence in linear

Example: $100 - 25 = 75$

Subtracting with multiple values in linear

Example: $100 - 25 - 50 = 25$

- When **subtracting** in professional, greater number goes on top
- You can only **subtract** 2 values at a time
- If the greater value is NOT first or on top, the value of the 2 digits will be negative
- When **subtracting** with **decimals**, align all **decimals** up and solve

Example: $25 - 100 = -75$

Multiplying

Multiplying in sequence in linear

Example: $2\times\ 50 = 100$

Multiplying multiple values in linear

Example: $2\ \times 2\ \times 8 = 32$

- When **multiplying** with multiple values in professional , every new value, in the result, insert a 0
- When **multiplying** with **decimals**, align **decimals** up, solve the question without **decimals**, then, for every digit before the **decimal**, is how many **decimal** places are in the result

Multiplication Chart (12 X 12)

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Dividing

Dividing in sequence in linear

Example: $50 \div 2 = 25$

Dividing multiple values in linear

Example: $50 \ \div 2 \ \div 5 = 5$

- **Divide** only 2 values at a time
- In professional, smaller number goes outside and the greater number goes inside leaving the value for the top
- When **dividing** with **decimals**, convert all the number to whole numbers, then divide

Integers

Adding integers

- When the negative **integer** is next to a positive symbol or **addition operator**, change it to a negative operator

Example: $5 + (-7) = 5 - 7 = -2$

- When there are 2 negative **integers**, **subtract** the two values together and you will end up with a negative result

Example: $-3 + (-4) = -3 - 4 = -7$

- When given several different **integers**, do it in order

Example: $-2 + (-4) + (-5) = -2 - 4 - 5 = -11$

Subtracting integers

- When a negative **operator** is next to a negative **integer**, the **integer** and **operator** both become positive

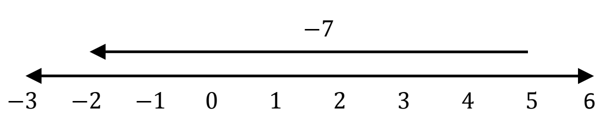
Example: $5 - (-3) = 5 + 3 = 8$

Example: $-8 - (-3) = -8 + 3 = -5$

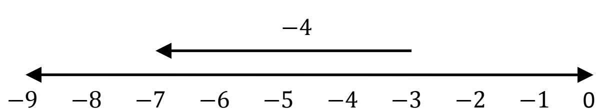
Example: $5 + (-4) - (-5) - 6 = 5 - 4 + 5 - 6 = 0$

Number Line

Effective way to add and subtract integers



$5 + (-7) = -2$



$-3 + (-4) = -7$

Multiplying and dividing integers

- When **multiplying** or **dividing integers**, there is a very simple rule to determine if the result will be negative or positive

$\times$ or $\div$	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

- The chart above shows that when **multiplying** 2 positive **integers** or 2 negative **integers**, the result is positive; while a positive and a negative **integer** have a negative result

Example: $-10 \times 2 = -20$

Example: $-10 \div (-2) = 5$

Example: $10 \div (-5) = -2$

Fractions

Using Fractions

- The **numerator** is the number on top and **denominator** is the number on the bottom
- The **numerator** is always the amount of the whole that is being taken up while the **denominator** tells you what the whole is out of

Formula: $\frac{\text{Numerator}}{\text{Denominator}}$

- **Fractions** are can be solved into **decimal** by **dividing** the **numerator** over **denominator**

Example in professional: $\frac{2}{5}$

Example in linear: $2/5$

Example in decimal: 0.4

- If the **denominator** is 1 and the numerator is a whole number, then the **fraction** in lowest terms is the **numerator**

Example: $\frac{4}{1} = 4$

- When you are trying to convert the **fraction** into lowest terms, be sure that whatever is done to the **numerator** is done to the **denominator**. Remember that a **numerator** or **denominator** can't be a decimal, they must be whole numbers

Example: $\frac{10}{2} = \frac{10/2}{2/2} = \frac{5}{1} = 5$

- If you ever require to convert a whole number in a **fraction**, remember how to identify a whole number

Example: $6 = \frac{6}{1}$

Reciprocals

- **Reciprocals** can be used on either **fractions** or numbers by using the opposite case
- If you want to find the **reciprocal** of a whole number, put the number over 1 (since a whole number, when **expressed** as a **fraction** is on top of 1, flip the 2 values, therefore the whole number becomes the **denominator**)

Example: $5 = \frac{5}{1} = \frac{1}{5}$

- If you want to find the **reciprocal** of a **fraction**, flip the **numerator** and **denominator**

Example: $\frac{2}{7} = \frac{7}{2}$

Adding Fractions

- Find the **lowest common denominator** (LCD) before multiplying all values in a **fraction** by a set digit, then the other **fraction** by a different digit

Example: $\frac{2}{5} + \frac{3}{10}$

- To solve, you must first **multiply** the first **fraction** by 2; **numerator** and **denominator**. Then you will get $\frac{2\times2}{5\times2} + \frac{3}{10} = \frac{4}{10} + \frac{3}{10}$ then simply **add** the **numerators** up to get $\frac{7}{10}$
- You always want to have common **denominators** when adding
- Only **add** the **numerators** and not the **denominators**
- Remember to always express in **lowest terms** by **dividing** the whole fraction by a set value
- Another way of getting the **LCD** is by using **prime factoring**, which means by finding the values of **denominators** through **multiplying prime numbers**. Start with **factors** of the first number then **add** any missing **factors** from the other number

Example: $\frac{1}{6}$ and $\frac{1}{8}$, LCD = 24

$6 = 2 \times 3, 8 = 2 \times 2 \times 2$

$\text{LCD} = 2 \times 3 \times 2 \times 2 = 24$

Subtracting Fractions

- Find a common **denominator** before **multiplying** all values in a **fraction** by a set digit, then the other **fraction** by a different digit. Then **subtract numerators**
- You always want to have common **denominators** when subtracting
- Only subtract the **numerators** and not the **denominators**

Example: $\frac{4}{6} - \frac{3}{5} = \frac{4\times5}{6\times5} - \frac{3\times6}{5\times6} = \frac{20}{30} - \frac{18}{30} = \frac{2}{30} = \frac{1}{15}$

Multiplying Fractions

- Simply **multiply** the **numerator** to the **numerator** and **denominator** to **denominator** regardless of the values

  Example: $\frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$

- Remember to always place in **lowest terms** by **dividing** all values by a set digit.

  Example: $\frac{6}{20} = \frac{3}{10}$

- You can also simplify your question by converting opposite **numerators** and **denominators** into lowest common numbers

  Example: $\frac{8}{9} \times \frac{3}{4} = \frac{2}{3} \times \frac{1}{1} = \frac{2}{3}$

Dividing Fractions

- Leave the first **fraction** along and then convert the second **fraction** to its **reciprocal**, then **multiply**

  Example: $\frac{2}{6} \div \frac{7}{8} = \frac{2}{6} \times \frac{8}{7} = \frac{16}{42} = \frac{8}{21}$

- You can also simplify your question by converting opposite **numerators** and **denominators** into lowest common numbers after the **reciprocal** is done

  Example: $\frac{2}{5} \times \frac{4}{9} = \frac{2}{5} \times \frac{9}{4} = \frac{1}{5} \times \frac{9}{2} = \frac{9}{10}$

Mixed Numbers

- A **mixed number** occurs when the **numerator** is greater than the **denominator**

Example: $\frac{21}{5}$

- To solve this, see how many times the **denominator** goes into the **numerator**, write the result before the **fraction** and leave the remainder where the **numerator** was with the same **denominator**

Example: $\frac{21}{5} = 4\frac{1}{5}$

- To convert a **mixed number** into a **fraction**, **multiply** the **denominator** by the whole number and add the **numerator**

Formula: $z\frac{x}{y} = (yz + x)$

Example: $4\frac{1}{5} = \frac{21}{5}$

Decimals

- A **decimal** less than 1 can become a **fraction**. Given that in **percent**, a number less than 1 is only out of 100, thus, any **decimal** given over 100 is a **fraction**. Then express in lowest terms

Example: $0.25 = \frac{25}{100} = \frac{1}{4}$

- To get a **decimal** from a fraction, **divide** the **numerator** by the **denominator**

Example: $\frac{1}{5} = 0.2$

Fractions, Decimals, Percents Conversions Chart

Fractions	Decimals	Percents	Fractions	Decimals	Percents
1	1.0	100%	1/6	0.16	16.6%
1/2	0.5	50%	1/8	0.125	12.5%
1/3	0.3	33.3%	1/10	0.1	10%
1/4	0.25	25%	2/3	0.6	66.6%
1/5	0.2	20%	3/4	0.75	75%

Percent

Fractions and **Percentages** are very similar. To find a **percentage** of something, **multiply** the percent to the number and **divide** by 100

Example: 25% of 300

$300 \times 25 \div 100 = 75$

- You can also convert the **percentage** into **percent** by making it less than one or dividing that value by 100

Example: 15% of 250

$250 \times 0.15 = 37.5$

- In a **pie chart**, you may want to find the **percent** of a section. When given the **angle**, you **divide** it by 360 and **multiply** by 100

Example: $90° \div 360° \times 100 = 25\%$

Ratios

Ratios are when you are comparing 1 thing to another

Example: $1 : 2$

- In this example, it is for everyone item, there is 2, therefore the **ratio** is 1 to 2
- You always want to express in lowest terms

Example: $4 : 6 : 16 = 2 : 3 : 8$

- In some cases, a question may give you a set of **ratios** and another with a missing value or values. Simply find what the alternative number was **multiplied** or **divided** by

Example: $4 : 6 : 8 = 2 : ? : 4$

$$\because \frac{4}{2} = 2; \frac{6}{2} = 3$$

- All that is required is one relation to be full

Example: $? : 5 : 10 = 5 : ? : 50$

$$10 \times 5 = 50;\ 5 \times 5 = 25;\ \frac{5}{5} = 1$$

$$\therefore 1 : 5 : 10 = 5 : 25 : 50$$

- Ratios can also be expressed as fractions by rearranging the **ratio**; **numerator** and **denominator**

Example: $3 : 6 = \frac{3}{6} = \frac{1}{2}$

- **Ratios** are also used for **probability** by comparing the likeliness of something against the total

Example: $4 : 6$

Exponents

Exponents can be **expressed** as a number to the **power of** or **square(s)**

Formula: $x^y,\ x = \text{Base}, y = \text{Exponent}$

Example: 2^3

- The example is **expressing** 2 to the **power of** 3, there for, 2 is **multiplied** by 2 three times.

Example in professional: $2^3 = 2 \times 2 \times 2 = 8$

Example in linear: 2^3

- When you have a negative **base**, there are two simple ways to solve
  - Write it in **expanded** form
  - If the **exponent** is even, the number is positive and vice-versa
- Ensure that the negative **exponent** is in **brackets**

Example: $(-3)^3 = (-3) \times (-3) \times (-3) = -27$

$-3^3 = -3 \times 3 \times 3 = -27$

Example: $(-4)^4 = 256$

$-4^4 = -256$

- When we have a negative **exponent**, we use the **reciprocal** of the number converting it into a **denominator** bringing the **exponent** with us, and making it positive

Example: $4^{-3} = \frac{1}{4^3} = 0.015625$

- Express as a **power of**

Example: Express as a power of 10 : $100 = 10^2$

Example: Express as a power of 2 : $128 = 2^7$

Exponents with Fractional Bases

- Simply write in **expanded** form

  Example: $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

- **Brackets** are necessary for otherwise the **exponent** only applies to either the **numerator** or **denominator**, not the whole **fraction** or **base**

  Formula: $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

- Negative **fractions** usually apply to the **numerator**

  Example: $\left(-\frac{2}{3}\right)^2 = \left(\frac{-2}{3}\right)^2 = \frac{4}{9}$

Multiplying and dividing exponents

- When **multiplying exponents**, we simply add the **exponents** together ONLY if the bases are equivalent

  Formula: $a^n \times a^m = a^{n+m}$

  Example: $5^2 \times 5^3 = 5^{2+3} = 5^5 = 3125$

- When **dividing exponents**, we simply **subtract** the **exponents** together ONLY if the bases are equivalent

  Formula: $a^n \div a^m = a^{n-m}$

  Example: $2^5 \div 2^3 = 2^{5-3} = 2^2 = 4$

Brackets and exponents

- When we have an **exponent** inside a **bracket** and an **exponent** outside the **bracket**, we **multiply** the 2 **exponents**

  Formula: $\left(a^b\right)^c = a^{bc}$

  Example: $(4^2)^3 = 4^{2\times3} = 4^6 = 4096$

Distributive property with exponents

- When there are 2 values inside a **bracket**, both with **exponents** and an **exponent** outside the **bracket**, the **exponent** outside, is **multiplied** to all.

Formula: $(ab)^c = (a^1b^1)^c = (a^{1c}b^{1c})$

Example: $(a^2b^3)^2 = (a^{2\times2}b^{3\times2}) = a^4b^6$

Example: $\frac{(2ab^2)^2\times(3a^3b^2)^2}{2ab^3} = \frac{(2^2a^2b^4)\times(3^2a^6b^4)}{2ab^3} = \frac{(4a^2b^4)\times(9a^6b^4)}{2ab^2} = \frac{36a^8b^5}{2ab^2} = 18a^7b^5$

- Anything raised to the power of 0 is 1

Formula: $x^0 = 1$

Example: $5^0 = 1$

- Keep in mind **Exponent Laws**

Example: $\left(\frac{6a^{-2}b^{-3}}{2a^2b^{-1}}\right)^{-2}$

$(3a^{-4}b^{-2})^{-2}$

$3^{-2}a^8b^4$

$\frac{a^8b^4}{9}$

Example: $\frac{(3x^3)(6xy^4)}{-9xy^2}$

$\frac{18x^4y^5}{-9xy^2}$

$-2x^3y^3$

Exponent Law

Law	Equation
Multiplication	$a^n \times a^m = a^{m+n}$
Division	$a^n \div a^m = a^{m-n}$
Power Law	$(a^n)^m = a^{n\times m}$
Power of a product	$(ab)^m = a^m b^m$
Power of a quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
Zero exponent	$a^0 = 1, a \neq 0$
Negative exponent	$a^{-1} = \frac{1}{a^n}, a \neq 0$

Square Roots

A **square root** is a **real number** which is **squared** to make its result

Formula: $\sqrt{x}$

Example: $\sqrt{25}$

$5^2 = 25 \therefore \sqrt{25} = 5$

- **Square roots** are the opposites of **squares** but when transferred, it carries both positive and negative operations

Formula: $\pm\sqrt{x}$

Example: $3x^2 = 432$

$x^2 = \frac{432}{3}$

$x = \pm\sqrt{144}$

$x = 12 \text{ or } -12$

- **Square roots** with **variables** with **exponents**; to solve, **square root** any **constant** and eliminate the **exponent**

Example: $\sqrt{4a^2} = 2a$

Squares and Square Roots

x	x^2	$\sqrt{x}$	x	x^2	$\sqrt{x}$
1	1	1	9	81	3
2	4	1.414	25	625	5
3	9	1.732	100	10000	10
4	16	2	1/2	1/4	0.707
5	25	2.236	1/4	1/16	1/2

- To break a **square root**, for **factoring** purposes, find 2 **terms** that multiply to create the original **square root**

Example: $x = \frac{2 \pm \sqrt{24}}{2}$

$$x = \frac{2 \pm \sqrt{4}\sqrt{6}}{2}$$

- When given a negative **term** within the **square root**, the result will always be inadmissible or rejected

Example: $\sqrt{-204} = \text{Inadmissable, rejected}$

Rational Exponents

Rational exponents are a combination of **square roots** and **exponents**.

- A **rational number** can be written in a **fraction**

Formula: $\left(\sqrt[n]{x}\right)^{m}; n = \text{nth root}, x = \text{radicand}, m = \text{exponent}$

- There are 2 forms

Example: $\left(\sqrt[n]{x}\right)^{\mathrm{m}} \rightarrow \text{Radical form}$

Example: $(x)^{\frac{m}{n}} \rightarrow \text{Exponent form}$

- These 2 forms will result in the same value
- Remember that a blank notation on a **square root** still has the **exponent** value of 2

Example: $\sqrt{9} = 3$

$(9)^{\frac{1}{2}} = 3$

Example: $\sqrt[3]{8} = 2$

$(8)^{\frac{1}{3}} = 2$

Example: $\sqrt[4]{16} = 2$

$(16)^{\frac{1}{4}} = 2$

- Basic conversion from **radical** form to **exponent** form

Formula: $\left(\sqrt[n]{x}\right)^{m} \rightarrow (x)^{\frac{m}{n}}$

- Working with negative **fractional exponents** requires conversion into positive to solve

Example: $7^{-\frac{1}{7}} = \left(\frac{1}{7}\right)^{\frac{1}{7}} = \sqrt[\frac{1}{7}]{7}$

- To solve a **radical**, the **numerator** must always be 1

Example: $5^{-\frac{3}{7}} \rightarrow \left(\frac{1}{5}\right)^{-\frac{3}{7}} = \left(\frac{1}{5}\right)^{3} \times \frac{1}{7} = \sqrt[7]{\left(\frac{1}{5}\right)^{3}} = \sqrt[7]{\frac{1}{125}}$

Example: $3^{\frac{2}{5}} \rightarrow 3^{2} \times \frac{1}{5} = \sqrt[5]{9}$

- **Algebra** and **rational exponents** work similar to the formula
- **Denominators** for on the **index** of the **radical** and the **numerator** is carried with the **radicand**

Example: $a^{\frac{3}{5}} = \sqrt[5]{a^3}$

- **Reciprocal** the **base** to get a positive number in the **exponent**

Example: $\left(\frac{25}{4}\right)^{-\frac{3}{2}} = \left(\frac{4}{25}\right)^{\frac{3}{2}} = \frac{4^{\frac{3}{2}}}{25^{\frac{3}{2}}} = \frac{\left(4^{\frac{1}{2}}\right)^3}{\left(25^{\frac{1}{2}}\right)^3} = \frac{2^3}{5^3} = \frac{8}{125}$

- Attempt to simplify when possible

Example: $\left(\frac{-27}{-8}\right)^{\frac{1}{3}} = \frac{3}{2}$

Example: $(\sqrt[3]{5^2})(\sqrt[3]{5}) = \left(5^{\frac{2}{3}}\right)\left(5^{\frac{1}{3}}\right) = 5^{\frac{3}{3}} = 5$

Example: $\left[(\sqrt{125})^4\right]^{\frac{1}{6}} = (\sqrt{125})^{\frac{4}{6}} = 125^{\frac{1}{3}} = 5$

Example: $\sqrt[3]{\sqrt{64}} \rightarrow \sqrt[3]{8} = 2 \rightarrow \left(64^{\frac{1}{2}}\right)^{\frac{1}{3}} = 64^{\frac{1}{6}} = \sqrt[6]{64} = 2$

Example: $(81a^8b^4)^{\frac{1}{4}} = 3a^2b$

- When simplifying, fist match the **bases** when working with more than 1 **polynomial**
- If the **bases** are the same, eliminate **bases** and solve for the **exponent**

Example:
$$3^{x+3} = 81$$
$$3^{x+3} = 3^4$$
$$x + 3 = 4$$
$$x = 4 - 3$$
$$x = 1$$

Example:
$$10^{2x+1} = 10000$$
$$10^{2x+1} = 10^3$$
$$2x + 1 = 3$$
$$2x = 2$$
$$x = 1$$

- Remember law of **exponents**

Example: $\frac{1}{\sqrt[3]{a}} = a^{-\frac{1}{3}}$

Example: $8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$

Example: $(-8)^{-\frac{5}{3}} = \frac{1}{\sqrt[3]{-8^5}} = -\frac{1}{32}$

Example: $\sqrt{\sqrt[5]{4a^4}} = \sqrt{4^{\frac{1}{5}}a^{\frac{4}{5}}} = 2^{\frac{1}{5}}a^{\frac{2}{5}}$

Example: $0.008^{-\frac{1}{3}} = \frac{1}{125^{\frac{1}{3}}} = \sqrt[3]{125} = 5$

Exponential Equations

Equations in which the **variables** are exponents. In order to solve, the **bases** must be the same for all **polynomials**.

- **Expand** the **equation** to solve
- Where the **bases** are the same, the **exponents** are equal

Formula: $x^m = x^n; m = n; a \neq -1, 0, 1$

Example:
$$2^{3x+4} = 4^{2x-5}$$
$$(2)^{3x+4} = (2^2)^{2x-5}$$
$$2^{3x+4} = 2^{4x-10}$$
$$3x + 4 = 4x - 10$$
$$x = 14$$

Example:
$$9^{-2x+1} = 27^{3x-2}$$
$$(3^2)^{-2x+1} = (3^3)^{3x-2}$$
$$3^{-4x+2} = 3^{9x-6}$$
$$-4x + 2 = 9x - 6$$
$$x = \frac{8}{13}$$

- Remember to follow law of **exponents**

Example:
$$2(4^{x+2}) = 1$$
$$\frac{\left(2(4^{x+2})\right)}{2} = \frac{1}{2}$$
$$4^x + 2 = \frac{1}{2}$$
$$(2^2)^x + 2 = 2^{-1}$$
$$2x + 4 = -1$$
$$x = -\frac{5}{2}$$

Example:
$$3^{x^2-2x} = 3^{x-2}$$
$$x^2 - 2x = x - 2$$
$$x^2 - 2x - x - 2 = 0$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$\therefore x = 2, x = 1$$

- **Common factor** when necessary

Example:

$2^{a+5} + 2^a = 1056$

$(2^a)(2^5) + 2^a = 1056$

$2^a(2 + 2^5) = 1056$

$2^a(66) = 1056$

$\frac{2^a(66)}{66} = \frac{1056}{66}$

$2^a = 16$

$a = 8$

Example:

$3^{g+3} - 3^{g+2} = 1458$

$(3^g)(3^3) - (3^g)(3^2) = 1458$

$3^g(3^3 - 3^2) = 1458$

$\frac{3^g(18)}{18} = \frac{1458}{18}$

$3^g = 81$

$3^g = 3^4$

$g = 4$

Example:

$2^{x+3} + 2^x = 288$

$(2^x)(2^3) + (2^x) = 288$

$2^x(1 + 2^3) = 288$

$\frac{2^x(9)}{9} = \frac{288}{9}$

$2^x = 32$

$2^x = 2^5$

$x = 5$

Rational Expressions

Expressions that include **variables** within **polynomials** and are **rational** (**fraction**).

- **Restrictions** are numbers that the **variable** cannot equal
- A **restriction** is made so that an answer will not equal 0
- Look for the **restriction** in the **factoring** step of the **expression** and the **denominator**
- Solve and state the **restriction**

Example: $\frac{24a^3b^2}{8ab}$

$3a^{2b}$

$\therefore a, b \neq 0$

Example: $\frac{a^2+4a}{a^2-4a}$

$\frac{a(a+3)}{a(a-4)}$

$\frac{a+3}{a-4}$

$\therefore a \neq 0, 4$

Example: $\frac{x^2-4}{5x+10}$

$\frac{(x-2)(x+2)}{5(x+2)}$

$\frac{x-2}{5}$

$\therefore x \neq -2$

Example: $\frac{1-4y^2}{8y^2-2}$

$\frac{(1-2y)(1+2y)}{2(4y^2-1)}$

$\frac{(1-2y)(1+2y)}{2(2y-1)(2y+1)}$

$\frac{-(1-2y)}{2(y-1)}$

$\therefore y \neq \frac{1}{2}, -\frac{1}{2}$

- Watch for **difference** of **squares** and **trinomial factoring**

Example: $\frac{x^2-8x+15}{x^2-25}$

$$\frac{(x-5)(x-3)}{(x-5)(x+5)}$$

$$\frac{(x-3)}{x+5}$$

$$\therefore x \neq 5, -5$$

Example: $\frac{6x^2-13x+6}{8x^2-6x-9}$

$$= \frac{(3x-2)(2x-3)}{(2x-3)(4x+3)}$$

$$= \frac{3x-2}{4x+3}$$

$$\therefore x \neq \frac{3}{2}, -\frac{3}{4}$$

Example: $\frac{2m^2-mn-n^2}{4m^2-4mn-3n^2}$

$$= \frac{(m-n)(2m+n)}{(2m-3n)(2m+n)}$$

$$= \frac{(m-n)}{2m-3n}$$

$$\therefore m \neq \frac{3n}{2}, -\frac{n}{2}$$

- Always watch for **common factors**

Example: $\frac{8y^2-10xy}{4y}$

$$\frac{2y(4y-5x)}{4y}$$

$$\frac{4y-5x}{2y}$$

$$\therefore y \neq 0$$

Multiplying and dividing rational expressions

- **Common factor**, **cross multiply**, **reduce**, and then **multiply**

Example:

$$\frac{8m^3}{3n^2}\times\frac{6n}{5m^2}$$
$$\frac{8m}{n}\times\frac{2}{5}$$
$$\frac{16m}{5n}$$
$$\therefore m,n\neq 0$$

Example:

$$\frac{15ab^2}{4c}\div\frac{8abc}{-3}$$
$$\frac{15ab^2}{4c}\times\frac{-3}{8abc}$$
$$\frac{15b}{4c}\times-\frac{3}{8c}$$
$$-\frac{45b}{32c^2}$$
$$\therefore\ a,b,c\ \neq 0$$

Example:

$$\frac{x^2-4}{x+3}\div\frac{4x-8}{3x+9}$$
$$\frac{(x+2)(x-2)}{x+3}\times\frac{3(x+3)}{4(x-2)}$$
$$\frac{3(x+2)}{4}$$
$$\therefore x\neq -3,2$$

Example:

$$\frac{x^2-xy-20y^2}{x^2-8xy+15y^2}\times\frac{x^2-xy-6y^2}{x^2+2xy-8y^2}$$
$$\frac{x^2+4xy-5xy-20y^2}{x^2-3xy-5xy-15y^2}\times\frac{x^2-2xy+3xy-6y^2}{x^2-2xy+4xy-8y^2}$$
$$\frac{x(x+4y)-5y(x+4y)}{x(x-3y)-5y(x-3y)}\times\frac{x(x-2y)+3y(x-2y)}{x(x-2y)+4y(x-2y)}$$
$$\frac{(x-5y)(x+4y)}{(x-5y)(x-3y)}\times\frac{(x+3y)(x-2y)}{(x+4y)(x-2y)}$$
$$\frac{x-5y}{x-5y}\times\frac{x-2y}{x-2y}$$
$$\frac{1}{1}\times\frac{1}{1}$$
$$1$$
$$\therefore x\neq 5y,3y,-4y,2y,0$$

Adding and subtracting rational expressions

- Find the **lowest common denominator**, then **add** or **subtract**, and **common factor** if possible

Example: $\frac{4x}{x+1}+\frac{6x}{x+1}$

$$\frac{4x+6x}{x+1}$$

$$\frac{10x}{x+1}$$

Example: $\frac{3a-b}{9}-\frac{a-2b}{3}-\frac{4a-3b}{6}, LCD = 18$

$$\frac{6a-2b-(6a-12b)-(12a-9b)}{18}$$

$$\frac{-12a+19b}{18}$$

Example: $\frac{2y+3}{3-4y}+\frac{5+2y}{4y-3}$

$$\frac{2y+3-(5+2y)}{3-4y}$$

$$\frac{2y+3-5-2y}{3-4y}$$

$$\frac{2}{3-4y}$$

- Always **common factor** and then find the **lowest common denominator**

Example: $\frac{3}{2m^{2n}}-\frac{1}{m^2n^3}+\frac{4}{5mn}, LCD = 10m^2n^3$

$$\frac{15n^2-10+8mn^2}{10m^2n^3}$$

Example: $\frac{x}{2x-4}-\frac{3}{3x-6}+1, LCD = 6(x-2)$

$$\frac{x}{2(2x-2)}-\frac{3}{3(x-2)}+\frac{1}{1}$$

$$\frac{3x-6+6(x-2)}{6(x-2)}$$

$$\frac{3x-6+6x+2}{6(x-2)}$$

$$\frac{9x-18}{6(x-2)}$$

$$\frac{9(x-2)}{6(x-2)}=\frac{9}{6}=\frac{3}{2}$$

Example: $\frac{2x}{x-2}+\frac{3x}{x+2}, LCD=(x-2)(x+2)$

$$\frac{2x(x+2)+3x(x-2)}{(x-2)(x+2)}$$

$$\frac{2x^2+4x+3x^2-6x}{(x-2)(x+2)}$$

$$\frac{5x^2-2x}{(x-2)(x+2)}$$

Example: $\frac{2x-1}{2x^2+3x+1}+\frac{2x+1}{3x^2+4x+1}, LCD=(2x+1)(3x+1)(x+1)$

$$\frac{2x-1}{(2x+1)(x+1)}+\frac{2x+1}{(3x+1)(x+1)}$$

$$\frac{(2x-1)(3x+1)+(2x+1)(2x+1)}{(2x+1)(3x+1)(x+1)}$$

$$\frac{6x^2-x-1+4x^2+4x+1}{(2x+1)(3x+1)(x+1)}$$

$$\frac{10x^2+3x}{(2x+1)(3x+1)(x+1)}$$

Example: $\frac{(x+3)(x+2)}{(x-2)(x-1)}\times\frac{x-1}{x+3}-\frac{6}{x+3}$

$$\frac{x+2}{x-2}-\frac{6}{x+3}$$

$$\frac{x+2}{(x-2)(x+3)}-\frac{6}{(x-2)(x+3)}$$

$$\frac{x^2+5x+6-6x+12}{(x-2)(x+3)}$$

$$\frac{x^2-x+18}{(x-2)(x+3)}$$

Example: $\frac{3x^2-5x-2}{3x^2+13x+4}\div\frac{x^2-x-2}{x^2+3x-4}$

$$\frac{(3x+1)(x-2)}{(3x+1)(x+4)}\times\frac{x^2+3x-4}{x^2-x-2}$$

$$\frac{(3x+1)(x-2)}{(3x+1)(x+4)}\times\frac{(x+4)(x-1)}{(x-2)(x+1)}$$

$$\frac{x-1}{x+1}$$

$$\therefore x\neq-\frac{1}{3},-4,2,-1,1$$

Entire and mixed radicals

- **Entire radicals** are **radicals** that are irrational
- **Mixed radicals** are **radicals** that sum to an **entire radical**
- Simplify **radicals** by finding **terms** that sum to the **entire radical**
- The **terms** must be **perfect squares**

Example: $\sqrt{40}$

$\sqrt{4} \cdot \sqrt{10}$

$2\sqrt{10}$

Example: $\sqrt{\frac{20}{9}}$

$\frac{\sqrt{20}}{\sqrt{9}}$

$\left(\frac{\sqrt{4}\sqrt{5}}{3}\right)$

$\frac{2\sqrt{5}}{3}$

- Keep in mind **like terms**

Example: $\sqrt{5}\sqrt{10}$
$\sqrt{50}$
$\sqrt{2}\sqrt{25}$
$5\sqrt{2}$

Example: $4\sqrt{3}\cdot 2\sqrt{7}$
$8\sqrt{21}$

Example: $2\sqrt{7}\cdot 3\sqrt{2}\cdot\sqrt{7}$
$6\sqrt{98}$
$6\sqrt{49}\sqrt{2}$
$6\cdot 7\sqrt{2}$
$42\sqrt{2}$

Example: $\frac{25-\sqrt{125}}{10}$
$\frac{25-5\sqrt{5}}{10}$
$\frac{5-\sqrt{5}}{2}$

- When you have a **negative radical**, the answer is indeterminable; however, mathematically expressed is the **imaginary number**. This is notated by i
- Ensure that **complex numbers** are being used

Example: $\sqrt{-80}$
$4i\sqrt{5}$

Example: $\sqrt{-1}$
i

Example: $\left(i\sqrt{3}\right)^2$
-3

Operating Radicals

Simplifying **radicals** by use of **factoring** and finding **like terms**. Also known as **complex numbers**

- **Adding** and **subtracting radicals** can be done through gathering **like terms**
- In this case, **like terms** are **terms** that have a common **root** in the **polynomial**
- Keep **roots** the same

Example: $4\sqrt{3} + 7\sqrt{20} - 5\sqrt{12} + 4\sqrt{5}$
$4\sqrt{3} + 7(2)\sqrt{5} - 5(2)\sqrt{3} + 4\sqrt{5}$
$4\sqrt{3} + 14\sqrt{5} - 10\sqrt{3} + 4\sqrt{5}$
$18\sqrt{5} - 6\sqrt{3}$

- When **multiplying radicals**, use distributive property and **multiply** the **roots** separately

Example: $7\sqrt{5}\left(3\sqrt{3} + 4\sqrt{2}\right)$
$21\sqrt{15} + 28\sqrt{10}$

- Always simplify when you can

Example: $-2\sqrt{3}\left(\sqrt{11} - \sqrt{6}\right)$
$-2\sqrt{33} + 2\sqrt{18}$
$-2\sqrt{33} + 2(3)\sqrt{2}$
$-2\sqrt{33} + 6\sqrt{2}$

Example: $\left(\sqrt{3} - 2\sqrt{2}\right)\left(\sqrt{3} + 2\sqrt{2}\right)$
$\sqrt{9} - 4\sqrt{4}$
$3 - 8$
-5

- You will have to **rationalize** the **denominator** in **fractions** because it is improper to have an **irrational denominator**
- **Rationalizing** means making a value **rational**
- To do so, **multiply** the **irrational** denominator to a **fraction** where both the **numerator** and **denominator** are equal. This is also known as **conjugating**

Formula: $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$

a, b, c, d

are always rational numbers.The **product** of **conjugating** is always **rational**

- **Multiply** accordingly and eliminate **roots** using **conjugates**

Example: $\frac{1}{\sqrt{3}}$

$$\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{\sqrt{3}}{3}$$

Example: $\frac{4}{3\sqrt{2}}$

$$\frac{4}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{4\sqrt{2}}{3(2)}$$

$$\frac{4\sqrt{2}}{6}$$

$$\frac{2\sqrt{2}}{3}$$

- When given 2 **polynomials** in the **denominator**, being **irrational**, invert the operator given and solve

Example: $\frac{-4}{6\sqrt{2}+2\sqrt{5}}$

$$\frac{-4}{6\sqrt{2}+2\sqrt{5}} \cdot \frac{6\sqrt{2}-2\sqrt{5}}{6\sqrt{2}-2\sqrt{5}}$$
$$\frac{-24\sqrt{2}+8\sqrt{5}}{35(2)-4(5)}$$
$$\frac{-24\sqrt{2}+8\sqrt{5}}{72-20}$$
$$\frac{-24\sqrt{2}+8\sqrt{5}}{52}$$
$$\frac{4\left(-6\sqrt{2}+2\sqrt{5}\right)}{4(13)}$$
$$\frac{-6\sqrt{2}+2\sqrt{5}}{13}$$

- Solve for the **variable** by putting in **standard form**

Example: $x = 3 \pm \sqrt{2}$

$$x - 3 = \pm\sqrt{2}$$
$$(x-3)^2 = 2$$
$$(x-3)^2 - 2 = 0$$
$$x^2 - 6x + 9 - 2 = 0$$
$$x^2 - 6x + 7 = 0$$

- When working with **roots** larger than 2, always simplify the **root** by finding perfect **roots**

Example: $\sqrt[3]{\sqrt{16}}$

$$\sqrt[3]{8}\sqrt[3]{2}$$
$$2\sqrt[3]{2}$$

Example: $\sqrt[3]{16} + \sqrt[3]{54}$

$$\sqrt[3]{8}\sqrt[3]{2} + \sqrt[3]{27}\sqrt[3]{2}$$
$$2\sqrt[3]{2} + 3\sqrt[3]{2}$$
$$5\sqrt[3]{2}$$

Statistics

Mean, **Median** and **Mode**, otherwise known as average, middle number and common value

Term	Definition and formula	Example
Mean (average)	Mean = sum of values/number of values	$2 + 4 + 6 = 12 \div 3 = 4$
Median	Middle number (in order), if between 2 numbers, then adjust value accordingly	$147, 148, 149, 150 = 148.5$
Mode	Number that appears most often	$4, 5, 5, 5, 6, 6, 7 = 5$
Range	The difference between the greatest and smallest number in the series	$33,37,33,31,41 = 10$

- An **outlier** is a measurement that differs significantly from the rest of the data

Example: $1,2,4,8,16,32, (33),128,256 \ldots$

Prime Numbers

Numbers that can only be divisible by 1 and themselves

- A **prime number** is a whole number with only 2 **factors**: itself and 1

  Examples: 2, 3, 5, 7, 11, 13, 17…

  Example: $3 = 3 \times 1$

- A **factor** is a number or array of numbers that are between the highest and lowest number

Prime Numbers Chart (1-100)

Grey: Prime Number

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime factoring

- When you are trying to find the **lowest common multiple** (**LCM**), you use **prime factoring**. By breaking down a number into the smallest **prime** digits

  Example: $28 = 2 \times 14 = 2 \times 2 \times 7$

- **Factor** trees are how a number can be broken down into **prime factors**

  Example:

$$512$$
$$2 \times 256$$
$$2 \times 2 \times 128$$
$$2 \times 2 \times 2 \times 64$$
$$2 \times 2 \times 2 \times 2 \times 32$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 16$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 8$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 4$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$
$$= 2^9$$

  Example:

$$36$$
$$12 \times 3$$
$$4 \times 3 \times 3$$
$$2 \times 2 \times 3 \times 3$$
$$= 2^2 \times 3^2$$

Composite numbers

- A **composite number** is a whole number with more than 2 **factors**; opposite rules of **prime numbers**

  Examples: $4, 6, 8, 9, 10, 12, 14 \ldots$

  Example: $6 = 3 \times 2$
  $6 = 6 \times 1$

The number 1

- The number 1 is neither a **prime** or **composite number**

Rational Numbers

A **rational number** is a number that can be written as a **quotient** (division question) of 2 **integers**, where the **divisor** is not 0

- A real number is also referred to as a **rational number**

  Examples: $-\frac{3}{5}; 0.25;\ -1\frac{3}{4};\ -3$

- There are many equivalent **rational numbers**

  Example: $-1\frac{1}{2} = -\frac{3}{2} = \frac{-3}{2} = \frac{3}{-2} = -1.5$

- Order of **rational numbers** (greatest to least or vice versa)

  Example: $-3, -2.55, -1\frac{1}{2}, 0.5, \frac{5}{4}, 2.5$

Order of Operation

The easiest way to remember the order of operations is through using an acronym

BEDMAS:	**Brackets**	( )
	Exponents	$^{2}\sqrt{}$
	Division and **Multiplication**	$\div \times$
	Addition and **Subtraction**	$+ \; -$

- In a question, we solve using BEDMAS; left to right

Example: $\frac{-3(4\times 2^2)^2+5-(-2)}{2^3} = \frac{-3(4^2\times 2^4)+7}{8} = \frac{-3(16\times 16)+7}{8} = \frac{-3(256)+7}{8} = \frac{-768+7}{8} = -\frac{768}{8} = -96$

Counter Example

A **counter example** is when a question believes that the information is true by displaying it through an example where the condition is true. A **counter example** is an example that disproves the belief of the question and shows clearly a false condition

- A **conjecture** is a general conclusion derived from apparent facts. A **conjecture** may not be true
- An **inference** is a conclusion based on reasoning and data
- A **counter example** can disprove a **conjecture** or hypothesis

System International (S.I.)

The system international is how we measure things metrically

Quantity	Base Unit	Symbol
Length	Meter	m
Mass	Gram	g
Volume	Litre/Cubic Meter	l/m^3
Time	Second	s

- Units can be **divided** or **multiplied** into **multiples** of 10 to give larger or smaller subunits
- Prefixes are used to indicate smaller or larger subunits

Common units used with the International System

Units of Measurement	Abbreviation	Relation
Metre	m	Length
Hectare	ha	G
Tonne	t	Mass
Kilogram	kg	Mass
Nautical mile	M	Distance (navigation)
Knot	kn	Speed (navigation)
Liter	L	Volume or capacity
Second	s	Time
Hertz	Hz	Frequency
Candela	cd	Luminous intensity
Degree Celsius	℃	Temperature
Degree Fahrenheit	℉	Temperature
Kelvin	K	Thermodynamic temperature
Pascal	Pa	Pressure/stress
Joule	J	Energy/work
Newton	N	Force
Watt	W	Power/radiant flux
Ampere	A	Electric current
Volt	V	Electric potential
Ohm	Ω	Electrical resistance
Coulomb	C	Electric charge

Metric system

Kilometre	**Hectometre**	**Decametre**	**Metre**	**Decimetre**	**Centimetre**	**Millimetre**
km	hm	dam	m	dm	cm	mm
$\mathbf{1000}$	100	10	1	$\frac{1}{10}; 0.1$	$\frac{1}{100}; 0.01$	$\frac{1}{1000}; 0.001$
$\mathbf{10^3}$	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

English system

Units of Measurement	**Abbreviation**	**Relation**
1 inch	in./”	
1 foot	ft.	12 inches
1 yard	yd.	3 feet
1 mile	mi.	1760 yards
1 square foot	sq. ft.	144 sq. inches
1 square yard	sq. yd.	9 sq. feet
1 acre	acre	4840 square yards $43,560 \text{ft}^2$
1 square mile	sq. mi.	640 acres
1 ton	T	2000 pounds
1 tablespoon		3 teaspoons
1 cup	c	16 tablespoons
1 pint	pt	2 cups
1 quart	qt	2 pints
1 gallon	gal	4 quarts
16 ounces	oz	1 pound
1 pound	lb	

Temperature conversion

Celsius to Fahrenheit	**Fahrenheit to Celsius**
$℃ \rightarrow ℉\text{: n} \times 1.8;\ +32$	$℉ \rightarrow ℃\text{: (n} - 32) \times 0.555$

Length and **area** conversion

Initial Unit	Second Unit	($1^{st} \rightarrow 2^{nd}$) Multiply	($2^{nd} \rightarrow 1^{st}$) Multiply
Centimetre	Inch	0.3937	2.54
Metre	Foot	3.2808	0.3048
Kilometre	Mile	0.6214	1.609
$Metre^2$	$Foot^2$	10.76	0.0929
$Kilometre^2$	$Mile^2$	0.3861	2.59

Weight and **volume** conversation

Initial Unit	Second Unit	($1^{st} \rightarrow 2^{nd}$) Multiply	($2^{nd} \rightarrow 1^{st}$) Multiply
Gram	Ounces	0.0353	28.35
Kilogram	Pound	2.2046	0.4536
Tonne	Ton	1.1023	0.9072
Millilitre	Ounces (fluid)	0.0338	29.575
Litre	Gallon	0.2642	3.785

Scientific Notation

Scientific notation is a method to express extremely large or small numbers

- Number of digits after the **decimal** determines the **exponent**
- Based on **powers of** 10

Example: $123\,000\,000\,000 = \text{number}$

$1.23\ \times 10^{11} = \text{scientific notation}1.23 = \text{coefficient}; 10 = \text{base}; 11 = \text{exponent}$

- The **coefficient** must be greater than or equal to 1 and less than 10
- The **base** must always be 10; a common notation for the base is the variable e

Example: $1.23\ e^{11}$

- **Decimal** place moves between the first and second digit
- When working with small numbers the **exponent** becomes negative
- Number of digits before the **coefficient** determines the **exponent** (negatively)

Example: $0.000\,000\,795$

$795 = \text{coefficient}7.93\ \times\ 10^{-7}$

- Negative **exponent** is referred to a **fraction**

Example: $10^{-7} = \frac{1}{10^7}$

Significant Digits

Digits that are significant when placed with multiple 0's. Significant digits are also known as significant figures, or Sig. Figs.

- All counted quantities are exact
- All measured quantities have some degree of error
- 0's placed before other digits are not significant

Example: $0.00(54)$ has 2 significant digits

- 0's placed between other digits are always significant

Example: 2.0036 has 5 significant digits

- 0's placed after other digits are significant only if there is a **decimal** place

Example: 1000.00 has 6 significant digits
1000 has 1 significant digit

- To change the number of a **significant digits** for a whole number (like 1000), convert it into **scientific notation**

Example: 1000 has 1 significant digit, but if changed into **scientific notation** it becomes 1.000×10^3 which gives this number 4 **significant digits**

- When **multiplying** and **dividing significant digits**, the result must have the same number of **significant digits** as the smallest measurement in the calculation

Example: $234.01 \times 2.50 = 585.025 \rightarrow 3$ significant digits $\therefore 585$

- When **adding** and **subtracting significant digits**, the result must have the same number of **significant digits** as the measurement with the least number of **decimal** points

Example: $2.1\text{cm} + 3.04\text{cm} + 1.02\text{cm} = 6.16\text{cm} \rightarrow 2$ significant digits $\therefore 6.2\text{cm}$

Rounding

Rounding numbers is the technique of shortening digit lengths to a easily understandable and realistic value.

- **Place value** is the classification of where a digit lies in a number
- Each classification is named after its base value, the beginning is the **decimal point**
- Digits that appear after the **decimal point** end with the suffix '-s' and start at one

Example: 1534
1: Thousands
5: Hundreds
3: Tens
4: Ones

- Digits that appear before the **decimal point** end with the suffix '-th' and start at ten

Example: 35.796
3: Tens
5: Ones
7: Tenth
9: Hundredth
6:Thousandth

- **Rounding** numbers is based on greatening the value of the higher digit place value
- Numbers between 0 and 4 are rounded down. Numbers between 5 and 9 are rounded up. When rounded up or down, the place value next to it increases, remains neutral, or decreases

Example: $\text{Number} = 2564; \text{Round to the nearest hundreds}$
$\because 6 \text{ (a tens place value) is} > 5 \text{ then}$
$\text{Number} = 2600$

Example: $\text{Number} = 0.872; \text{Round to the nearest hundredth}$
$\because 2 \text{ (a thousandth place value) is} < 5 \text{ then}$
$\text{Number} = 0.870$

Algebra

Polynomials

A polynomial is a combination of **constants** and **variables** that are bound by **multiplication** and **division**. There are 4 classifications of polynomials. They are a monomial, binomial, trinomial and polynomial. Each indicates whether there is 1, 2, 3 or more than 3 **terms**.

Type	Number of Terms	Examples
Monomial	1	$5y, 3a, 2x, 50, x, xy, xyz$
Binomial	2	$5y + 3a, 2x + 10, 50x + 3a, x + y, p^2 + p$
Trinomial	3	$p^3 + p^2 + p, 2^3 + 2^2 + 2, 2y + 4k - 7z$
Polynomial	$3 +$	$a^2b + 5 - 4ab + 2$

- A **Monomial** is a number, a product of one or more **variables**, or the product of a number and or more **variables**. The **coefficient** is the number part of a **monomial**

  Example: $bx; b = \text{coefficient}, x = \text{variable}, bx = \text{monomial}$

- A **polynomial** is formed by adding or subtracting monomials. Each monomial is a term of the polynomial. Some polynomials have special names
- **Monomial**: $6x, -3x^2, 4x3y3$
- **Binomial**: $3x + y, 2x + 7, 6x^2 - 2xy$
- **Trinomial**: $x^2 + xy + y, 6x^2 - 3b^2c^2 + 2abcd$
- **Polynomials** with more than 3 terms are called polynomials.
- **Polynomials** can also be classified by the degree of the variable
- The **Degree** is the highest number of the sum of the exponents

Example:

Polynomial	Term	Name	Degree
$-6x$	1	Monomial	1
2	1	Monomial	0
$6x - 7y$	2	Binomial	1
$5x^3 + x^2 - 7x + 2$	4	Polynomial	3

- How each are classified are through the number of **terms** used
- A **term** is classified by **constants** and **variables** that are not bound by **addition** or **subtraction operators**

Example: $2a = 1 \text{ terms} = \text{monomial}$

Example: $2a + 3a = 2 \text{ terms} = \text{binomial}$

Example: $2a - a + 4b = 3 \text{ terms} = \text{trinomial}$

Example: $2a - a^3 + 5/4 - ab = 4 \text{ terms} = \text{polynomial}$

- A **variable** is a letter that represents a value

Examples: $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z$

- A **coefficient** is a number that is **multiplied** by a **variable**

Examples: $3x \rightarrow 3,\ x \rightarrow 1$

- A **constant** is a term that doesn't include any **variables** (a number by itself)

Example: $3x + 50 \rightarrow 50$

- A **degree of a term** means the **sum** of the **exponents** on the **variables** in a **term**

Examples: $x^2 \rightarrow 2, 3y^4 \rightarrow 4, (-2)a^3b \rightarrow 4, (-3) \rightarrow 0$

- A **degree of a polynomial** means the degree of the highest **term**

Example: $x - 2 \rightarrow 1$

Example: $3w^2 - 2w + 5 \rightarrow 2$

When **graphing** with **variables**, there are 2 types of **variables**. **Independent variables** and **dependant variables**

- **Independent variables** are **variables** that are not affected by other **variables** (x axis)
- **Dependant variables** are **variables** that can be affected by other **variables** (y axis)
- A **dependant variable** is a **variable** affected by another **variable**. On a **graph**, the **dependant variable** is labelled on the y axis
- An **independent variable** is a **variable** that affects other **variables**. On a **graph**, the **independent variable** is labelled on the x axis

Collecting Like Terms

Like terms is a way of simplifying a question by taking terms with similar **variables**

- In order to **collect like terms**, the **terms** that are being collected must have the same **variable**/letter and the same **exponent**
- **Add** or **subtract terms** from each other
- Rewrite in greatest to least **term** and in alphabetical order

Example: $3x + 5x = 8x$

Example: $2x + 3 + 3x + 6 = 2x + 3x + 3 + 6 = 5x + 9$

Example: $2x + 7x + 3x + 4z + 5 + 2z + 3x + 1 = 3x + 3x + 2x + 4z + 2z + 7 + 5 + 1 = 8x + 6z + 13$

Example: $3x + 2y + 6x - y + 3 = 6x + 3x + 2y - y + 3 = 9x + 1y + 3 = 9x + y + 3$

- When there are **variables** with different **exponents**, only group same **exponent variables** together, not all **variables**

Example: $5x^2 - 3 + 2x - 2x^2 + x - 6 = 5x^2 - 2x^2 + 2x + x - 6 - 3 = 3x^2 + 3x - 9$

Add polynomials

- Remove **brackets** and **collect like terms**

Example: $(k+3) + (3k-4) = k+3+3k-4 = 3k+k+3-4 = 4k-1$

Example: $(6x^2 + 3x - 5) + (7x^2 - 3x - 10) = 6x^2 + 3x - 5 + 7x^2 - 3x - 10 = 7x^2 + 6x^2 + 3x - 3x - 5 - 10 = 13x^2 - 15$

Example: $(p+3) + (-2p+1) = p + 3 + (-2p) + 1 = p + (-2p) + 3 + 1 = (-p) + 4$

Subtract polynomials

- **Add** opposite, open **brackets** then **collect like terms**
- Switch opposite for every **minus sign** before a **bracket**
- When there is a negative sign outside a **bracket**, think of it as negative 1 and use d**istributive property** to get the opposite

Example: $(3t+5) - (-7t+1) = (3t+5) + (7t-1) = 3t + 5 + 7t - 1 = 3t + 7t + 5 - 1 = 10t + 4$

Example: $(2k^2 - 6k + 8) - (5k^2 - 6k + 8) = (2k^2 - 6k + 8) + (-5k^2 + 6k - 8) = 2k^2 - 6k + 8 - 5k^2 + 6k - 8 = 2k^2 - 5k^2 - 6k + 6k + 8 - 8 = (-3k^2)$

Multiply polynomials

- **Multiplying monomials**
- Remove **brackets**
- **Multiply coefficients**
- When **multiplying variables**, **collect like terms** of the similar **variable** and per **variable**, **add** an additional **exponent**
- **collect like terms**

Example: $(9x^2)(3x) = 27x^3$

Example: $(-8xy)(6x^2y^4z) = -48x^3y^5z$

- **Multiplying Binomials**
- Similar to **multiplying monomials** but there is a specific order

Formula: **FOIL: First term, outside term, inside term, last term**

Example: $(x+3)(x-7)$
$x \times x$
$x \times -7$
$3 \times x$
3×-7
$= x^2 - 7x + 3x - 21$
$= x^2 - 4x - 21$

Example: $(6y-3)(2x+7)$
$= 12xy + 42y - 6x - 21$
$= 12xy - 6x + 42y - 21$

Example: $6(3x-4)(x-7)$
$= 6(3x^2 - 21x - 4x + 28)$
$= 6(3x^2 - 25x + 28)$
$= 6(3x^2) + 6(-25x) + 6(28)$
$= 18x^2 - 150x + 168$

Simplify: $8 - 3(4x-3)(5x-2) - (3x+5)(2x+5)$
$= 8 - 3(20x^2 - 8x - 15x + 6) - (6x^2 + 15x + 10x + 25)$
$= 8 - 3(20x^2 - 23x + 6) - (6x^2 + 25x + 25)$
$= 8 - 3(20x^2) - 3(-23x) - 3(6) - (6x^2 + 25x + 25)$
$= 8 - 60x^2 + 69x - 18 - 6x^2 - 25x - 25$
$= -66x^2 + 44x - 35$

Squaring Binomials

- **Multiply** the bracket **term** by how ever many **exponents** there are

Example:
$(x-2)^2$
$= (x-2)(x-2)$
$= x^2 - 2x - 2x + 4$
$= x^2 - 4x + 4$

Example:
$(x-2y)^2$
$= (x-2y)(x-2y)$
$= x^2 - 2xy - 2xy + 4y^2$
$= x^2 - 4xy + 4y^2$

Example:
$(6x-4y)^2$
$= (6x-4y)(6x-4y)$
$= 36x^2 - 24xy - 24xy + 16y^2$
$= 36x^2 - 48xy + 16y^2$

Example:
$(3x+2y)^2$
$= (3x+2y)(3x+2y)$
$= 9x^2 + 6xy + 6xy + 4y^2$
$= 9x^2 + 12xy + 4y^2$

- The **product** of **sum difference** can be solved 2 ways
- Simplify

Example: $(x - y)(x + y)$
$= x^2 + xy - xy - y^2$
$= x^2 - y^2$

Example: $(3x + 4y)(3x - 4y)$
$= 9x^2 - 12xy + 12xy - 16$
$= 9x^2 - 16y^2$

- Watch for the **operators**

Formula: $(a + b)(a - b) = a^2 - b^2$

Formula: $(a + b)^2 = a^2 + 2ab + b^2$

Formula: $(a - b)^2 = a^2 - 2ab + b^2$

- Simplify

Example: $(5x + 2y)(5x - 2y)$
$= (5x)^2 - (2y)^2$
$= 25x^2 - 4y^2$

Example: $16y^2 - 25x^2$
$= (4y - 5x)(4y + 5x)$

- Look for **common factors** first

Example: $18x^2 - 8y^2$
$= 2(9x^2 - 4y^2)$
$= 2(3x - 2y)(3x + 2y)$

Example: $\frac{x^2}{4} - \frac{1}{9}$
$= \left(\frac{x}{2} - \frac{1}{3}\right)\left(\frac{x}{2} - \frac{1}{3}\right)$

Example: $3(2x + 3)^2 - (2x - 4)(2x + 4)$
$= 3(2x + 3)(2x + 3) - (4x^2 - 16)$
$= 3(4x^2 + 12x + 9) - 4x^2 + 16$
$= 12x^2 + 36x + 27 - 4x^2 + 16$
$= 8x^2 + 36x + 43$

Perfect square trinomials

- First and last **terms** are perfect **squares**
- Middle **term** is twice the **product** of the **square roots** of the first and last **terms**

Formula: $a^2 + 2ab + b^2 = (a + b)^2$

Formula: $a^2 - 2ab + b^2 = (a - b)^2$

- **Perfect squares** can be determined through a second method

Example: $4x^2 + 12x + 9$
$\therefore 2(\sqrt{4}\sqrt{9}) = 12$

- Simplify

Example: $x^2 + 6x + 9$
$= (x + 3)^2$

Example: $x^2 - 10x + 25$
$= (x - 5)^2$

Example: $9x^2 + 12xy + 4y^2$
$= (3x + 2y)^2$

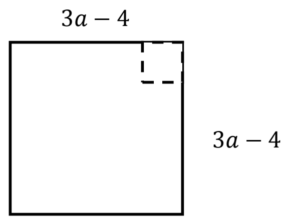
Example: $x^3 - 18x^2 + 91x$
$= x(x^2 - 18x + 81)$
$= x(x - 9)^2$

Example: $32x^2 - 8$
$8(4x^2 - 1)$
$8(2x + 1)(2x - 1)$

Example: $9x^2 - 6x + 1$
$(9x - 1)^2$

Example: $x^2 - 36$
$(x + 6)(x - 6)$

Example: The **area** of a **square** is represented by
$A = 9a^2 - 24a + 16$, for a being a positive number. Find a
$A = (3a - 4)^2$
$A = (3a - 4)(3a - 3)$
$3a - 4 > 0$
$3a > 4$
$a = \frac{4}{3}$



- **Perfect squares** can also be found in other **polynomials**
- Watch for **exponents** and **perfect squares**

Example: $x^4 - 81$
$= (x^2 + 9)(x^2 - 9)$
$= (x^2 + 9)(x - 3)(x + 3)$

- The **variable** may have more than 1 answer

Example: $2x^2 + 7x = -3$
$2x^2 + 7x + 3 = 0$
$2x^2 + x + 6x + 3 = 0$
$x(2x + 1) + 3(2x + 1) = 0$
$(x + 3)(2x + 1) = 0$
$\therefore x = -3, x = -\frac{1}{2}$

- Simplify

Example: $x^2 - 4x = -4$
$x^2 - 4x + 4 = 0$
$(x - 2)^2 = 0$
$x - 2 = 0$
$x = 2$

Example: $x^2 + 49$
Can't be factored

Divide polynomials

- Remove **brackets**
- **Divide coefficients**
- When **dividing variables**, **collect like terms** of the similar **variable** and per **variable**, **subtract** an additional **exponent**
- **collect like terms**

Example: $\frac{36x^5y^4z^3}{2x^3y^{-7}z^2} = 18x^2y^{11}z$

- **Expand** and **simplify**; 2 methods
- Second method involves **dividing** the 2 **simplified terms**

Example: $-6x(x-3) + 5x(x-7)$

$= -6x^2 + 18x + 5x^2 - 35x = -x^2 - 17$

Second Method

$= \frac{-6x^2 + 18x}{5x^2 - 35x} = -x^2 - 17x$

Difference of Squares

- A simple way of **common factoring**
- Only applies to a **difference**

Formula: $(x^2 - b)$ Where b is a perfect square
$\left(x + \sqrt{b}\right)\left(x - \sqrt{b}\right)$

Example: $(x^2 - 25)$
$(x + 5)(x - 5)$

Distributive Property

When you have a **constant** or **variable** outside a **bracket**, you distribute the **constant** or **variable** to every **term** within the **bracket** and **multiply** each **term** by that **variable** or **constant**. Also known as **expanding** or **simplifying**

Formula: $a(x+y)=ax+ay$

Example: $7(x-3)=(7x)+\big(7(-3)\big)$
$=7x-21$

Example: $6(p+q)+2x(p+q)$
$=(p+q)(6+2x)$

- When there is a **term** with a **variable** outside and inside the **bracket**, the **variable** is raised to the **power of** that **variable** or the **sum** of the **exponents**

Example: $(7y-1)(5y)=\ 5y(7y-1)=\ 5y(7y)+5y(-1)=\ 35y^2-\ 5y$

Example: $8x^2+y+4xy+x$
$=8x^2+x+4xy+y$
$=x(8x+1)+y(4x+1)$

- Dealing with **fractions** is no different, it applies as a **term**

Example: $\frac{1}{2}(2w-6)=\frac{1}{2}(2w)+\frac{1}{2}(-6)=w-3$

- With **variables**, remember with like **variables multiplied** with each other makes it raised to the **power of** the **sum** of the **exponents**

Example: $x(x+4)+2x(x+1)=x(x)+x(4)+2x(x)+2x(1)=x^2+4x+2x^2+$
$3x=2x^2+x^2+4x+2x=3x^2+6x$

Factoring

Factoring is the opposite of **expanding**. **Factoring** is used to confirm what the **Greatest Common Factor** (**GCF**) is. We can easily assume what the **GCF** is but then we must confirm it. To find the **GCF**, find a **term** or **constant** or **variable** or both that fits all the **terms**

- In a **polynomial** with **variables** and **constants**, we identify the **GCF**, and then put the **polynomial** inside **brackets** and the **GCF** before the **brackets**. We **divide** each term in the **brackets** by the **GCF**

  Example: $3x + 6 \text{ GCF } = 3, 3(x + 2)$

- With **exponents**

  Example: $2x + 8x^2 \text{ GCF } = 2x, 2x(1 + 4x)$

  Example: $3x^2 + 2x + xy \text{ GCF} = x, x(3x + 2 + y)$

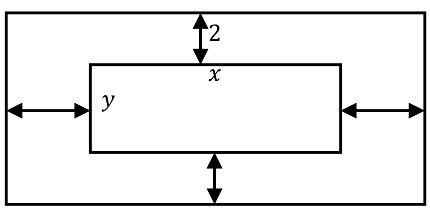
  Example: $5m^2t - 10m^2 - t^2 - 2t$
  $= 5m^2(t - 2) + t(t - 2)$
  $= (t - 2)(5m^2 + t)$

- With **variables** only

  Example: $b^5u^2m - b^3um^2 \text{ GCF } = b^3um, b^3um(b^2u - m)$

- Simply try to find what fits all **terms**
- Find the area in **factored** form

  Example:



$A = (x + 2 + 2)(y + 2 + 2) - xy$
$= (x + 4)(y + 4) - xy$
$= xy + 4x + 4y + 16 - xy$
$= 4x + 4y + 16$
$= 4(x + y + 4)$

Factor by grouping

- Some **polynomials** do not have common **factors** in all of their terms. These **polynomials** can sometimes be factored by grouping terms that do not have a common **factor**
- Group **terms** that have a common **factor**

Example: $ax + ay + bx + by$
$= a(x + y) + b(x + y)$
$= (x + y)(a + b)$

- **Factoring** a **trinomial**

Formula: $x^2 + bx + c$

- b and c and **constants**
- To find the grouped **factor**, you must find 2 **integers** which will equal the **sum** of b and the **product** of c

Example: $x^2 + 8x + 15$
What 2 numbers add to 8 and multiply to 15; $(3, 5)$
$3 + 5 = 8$
$3 \times 5 = 15$
$\therefore (x + 3)(x + 5)$

- Watch for the **constants**

  Example: $x^2 - x - 30; (-6,5)$
  $= (x-6)(x+5)$

- Watch for the **exponents**

  Example: $x^3 + 18x^2 + 72x; (12,6)$
  $= x(x^2 + 18x + 72)$
  $= x(x+12)(x+6)$

- Remember that the rule applies to the whole **term**

  Example: $(x-y)^2 - 5(x-y) + 6; (-3,-2)$
  $= (x-y-3)(x-y-2)$

- When there are multiple **variables**, find the 2 **integers** and include the alternate **variable term** beside the **constants**

  Example: $x^2 + 14xy - 32y^2; (16,-2)$
  $= (x+16y)(x-2y)$

- In some cases the **polynomial** will not be able to **factor**

  Example: $x^2 - 5x - 2$
  Can't be factored

- When x has a **constant multiple**, it is considered a **quadratic equation**
- A **quadratic equation** is a **polynomial equation** of the second **degree**

Formula: $ax^2 + bx + c, a \neq 1$

- To find the grouped **factor**, you must find 2 **integers** which will equal the **sum** of b and the **product** of a and c

Example: $2x^2 - x - 6$; add: -1, multiply: -12; $(-4,3)$
$= 2x^2 - 4x + 3x - 6$
$= 2x(x - 2) + 3(x - 2)$
$= (x - 2)(2x + 3)$
Check:
$(x - 2)(2x + 3)$
$= 2x^2 + 3x - 4x - 6$
$= 2x^2 - x - 6$

- Watch for the **exponents**

Example: $6x^2 + xy - 2y^2$; add: 1, multiply: -12; $(-3,4)$
$= 6x^2 - 3xy + 4xy - 2y^2$
$= 3x(2x - y) + 2y(2x - y)$
$= (2x + 2y)(2x - y)$

- Look for **common factors**
- **Common factors** with **constants**

Example: $4x^2 + 4xy - 8y^2$; add: 4, multiply: -32; $(8, -4)$
$= 4x^2 - 4xy + 8xy - 8y^2$
$= 4x(x - y) + 8y(x - y)$
$= (4x + 8y)(x - y)$
$= 4(x + 2y)(x - y)$
Common factors
$= 4(x^2 + xy - 2y^2)$; add: 1, multiply: -2; $(2, -1)$
$= 4(x + 2y)(x - y)$

- **Common factors** with **variables**

Example: $2m^2 + 7m^2 - 30m$; add: 7, multiply: -60; $(12, -5)$
$= m(2m + 7m - 30)$
$= m(2m^2 + 12m - 5m - 30)$
$= m(2m - 5)(m + 6)$

- **Rearranging** the question will sometimes make it easier to solve and then finding **common factors**

Example: $15n^2 - n - 2; (-6,5)$
$= 15n^2 - 6n + 5n - 2$
$= 15n^2 + 5n - 6n - 2$
$= 5n(3n + 1) - 2(3n + 1)$
$= (5n - 2)(3n + 1)$

- In some cases the **polynomial** will not be able to **factor**

Example: $5x^2 + 9x + 2; \text{add: } 9, \text{ multiply: } 10$
can't be factored

- There are scenarios in which there are multiple numbers which **add** and **multiply** to a **term**

Example: For what value of k can this trinomial be factored?
$3x^2 + kx + 5; \text{add: } k, \text{ multiply: } 15;$
$(3,5) = 8, (-3,-5) = -8, (-15,-1) = -16, (15,1) = 16$
$= k \in \{-16,-8,8,16\}$

Solving one step Equations

Adding and subtracting

- Isolate your **variable**
- Whatever is done to one side must be done to the other side

Example:
$x + 3 = 5$
$x + 3 - 3 = 5 - 3$
$x = 2$

Example:
$x - 3 = -2$
$2 - 3 + 3 = -2 + 3$
$x = 1$

Multiplying and dividing

- Isolate your **variable**
- Whatever is done to one side must be done to the other side

Example:
$4x = 20$
$\frac{4x}{4} = \frac{20}{4}$
$x = 5$

- Always keep your **variable** positive

Example:
$-k = 11$
$-1k = 11$
$-1k = \frac{11}{-1}$
$k = -11$

Solving two step Equations

Recall the **order of operations** (**BEDMAS**), solve equations in reverse order of **BEDMAS**; **SAMBED**

- Isolate your **variable** or **term**
- Whatever is done to one side must be done to the other side

Example:

$$\boxed{3x} + 12 = 15$$
$$3x + 12 - 12 = 15 - 12$$
$$3x = 3$$
$$\frac{3x}{3} = \frac{3}{3}$$
$$x = 1$$

Example:

$$-2x - 6 = 8$$
$$-2x = 8 + 6$$
$$x = \frac{14}{-2}$$
$$x = -7$$

Solving multi step Equations

Solve

- Isolate your **variable**
- Whatever is done to one side must be done to the other side
- Get all **variable terms** and **constant terms** to separate side
- Use reverse order of **order of operations**

Example:

$$7y = 2(y + 15)$$
$$7y = 2y + 30$$
$$7y - 2y = 30$$
$$5y = 30$$
$$y = \frac{30}{5}$$
$$y = 6$$

- When trying to check work, simply substitute your answer into the question and solve separately for both sides. If the results of both sides are equivalent, then your answer is correct

Example: (refer to above example)

$$7y = 2(y + 15)$$
$$7(6) = 2(6 + 15)$$
$$42 = 2(21)$$
$$42 = 42$$
$$\therefore y = 6$$

Solving Equations with Fractions

- With one **fraction**, **multiply** the **denominator** to each **term** and/or **bracket expression**
- Do not distribute the **fraction**, you want to negate the **fraction**
- Always have the **variable** on the left side

Example:

$$6 = \frac{1}{3}(8 + x)$$
$$3(6) = 3\left[\frac{1}{3}(8 + x)\right]$$
$$18 = 8 + x$$
$$18 - 8 = x$$
$$x = 10$$

- With **multiple fractions**, find the **lowest common denominator LCD** and **multiply** the **LCD** to each **term** and/or **bracket expression**
- Once you have done so, **divide** the **denominator** in the **fraction** by the **LCD**. Eliminate the **fraction** and **multiply** the **quotient LCD** by the **term** and/or **bracket expression**
- Then use **distributive property** once no **fraction** remains

Example:

$$\frac{k+2}{3} = \frac{k-4}{5}$$
$$15\left(\frac{k+2}{3}\right) = 15\left(\frac{k-4}{5}\right)$$
$$5(k + 2) = 3(k - 4)$$
$$5k - 3k = -12 - 10$$
$$2k = -22$$
$$k = -\frac{22}{2}$$
$$k = -11$$

- If a **fraction** is a numerator less than 1, then, with the **LCD**, you **divide** the **LCD** with the **denominator** and **multiply** the **quotient** with the **numerator**

Example: $\frac{3}{4} = 4\left(\frac{3}{4}\right) = 1(3) = 3$

Rearranging Formulas

Single step

- Isolate the **variable** you want or **term** with **variable**
- Keep the isolated **variable** on the left side

Example:

$$d = \boxed{a} + b$$
$$d - b = a + b - b$$
$$d + b = a$$
$$a = d - b$$

Example:

$$c = 2\pi^{\boxed{r}}$$
$$\frac{c}{2\pi} = \frac{2\pi^{r}}{2\pi}$$
$$\frac{c}{2\pi} = r$$
$$r = \frac{c}{2\pi}$$

Example:

$$A = \boxed{s}^2$$
$$\sqrt{A} = \sqrt{s^2}$$
$$\sqrt{A} = s$$
$$s = \sqrt{A}$$

Multi step

- Isolate the **variable** you want
- Keep the isolated **variable** on the left side
- Use reverse order of **order of operations**

Example:

$$y = m\boxed{x} + b$$
$$y - b = mx$$
$$\frac{y - b}{m} = x$$
$$x = \frac{y - b}{m}$$

Word Problems

Let and therefore statements

- **Let statements** defines a **variable**
- **Therefore statement** justifies the **answer**

Example: A number plus 3 is 8. What is the number?
Let x represent the number

$x + 3 = 8$
$x + 3 - 3 = 8 - 3$
$x = 5$
$\therefore x = 5$ Or therefore the number is 5

- Be careful of the wording

Example: A number 3 less is 8. What is the number?
Let x represent the number

$x - 3 = 8$
$x - 3 + 3 = 8 + 3$
$x = 11$
$\therefore x = 11$

Consecutive numbers

- **Consecutive numbers** are **integers** that come one after the other without skipping.

  Example: $1, 2, 3$ or $-1, 0, 1$ or $43, 44, 45$

- Use extended **let statements** to define the **variable** along with other numbers by using the **variable** in the statement
- **Collect like terms**
- Use **If** and **Then statements** to justify your answer and **variables**
- End with a **therefore statement**

Example: The **sum** of 3 **consecutive numbers** is 33

Let x represent the first number, then $x+1$ represent the second number and $x+2$ represent the third number

$x+(x+1)+(x+2)=33$

$x+x+1+x+2=33$

$3x+3=33$

$3x+3-3=33-3$

$\frac{3x}{3}=\frac{30}{3}$

$x=10$

If $x=10$ Then,

$x+1=10+1=11$

And $x+2=10+2=12$

∴ The 3 **consecutive numbers** are $10, 11, 12$

- **Consecutive EVEN or ODD numbers** are **integers** that come evenly or odd in series

Example: $2, 4, 6$ or $-3, -1, 1$ or $43, 45, 47$

- Same rules apply but in this case, be sure to adjust the statements and **variables** accordingly

Example: The **sum** of 3 **consecutive even numbers** is 18
Let x represent the first number, then $x + 2$ represent the second number and $x + 4$ represent the third number

$x + x + 2 + x + 4 = 18$
$3x + 6 - 6 = 18 - 6$
$\frac{3x}{3} = \frac{12}{3}$
$x = 4$

If $x = 4$ Then,
$x + 2 = 4 + 2 = 6$
And $x + 4 = 4 + 4 = 8$
∴ The 3 **consecutive numbers** are $4, 6, 8$

- When word problems come in more complex orders, work backwards

Example: The length of a rectangle is 2 more than twice the width. If the perimeter is 40m, what are the dimensions?
Let w represent the width, then $2w + 2$ represent the length

$p = 2(lw)$
$40 = 2(2w + 2 + w)$
$40 = 2(3w + 2)$
$40 = 2(3w) + 2(2)$
$40 = 6w + 4$
$40 - 4 = 6w + 6 - 6$
$\frac{36}{6} = \frac{6w}{6}$
$w = 6$

If $w = 6$ Then,
$2w + 2 = 2(6) + 2 = 14$
∴ The dimensions are 6m x 14m

Example: Pablo is 7 years older than Mario. The sum of their ages is 13. What are their ages?

Let m represent Mario's age, then $m+7$ represent Pablo's age

$m+m+7=13$
$2m+7=13$
$2m+7-7=13-7$
$\frac{2m}{2}=\frac{6}{2}$
$m=3$

If $m=3$ Then,
$m+7=3+7=10$
$\therefore$ Pablo's age is 10 and Mario's age is 3

Series and Sequences

A **sequence** is a set of numbers in order

- **Sequences** can be **finite** (terminate), or **infinite** (never ending) separated by commas

Example: $5, 6, 7, 8$ … (**Infinite**)
$4, 7, 10, 13$ (**Finite**)
$2, 4, 5, 8, 10$ … (**Infinite**)

- Each number in a **sequence** is called a **term**
- Each **term** can be denoted by t_n or $f(n)$ where n is the number position in the **sequence**
- The **sequence** can be defined by a formula

Example: t_1 is the first **term**
t_2 is the second **term**
t_n is the nth or **general term**

- When given a formula, you can solve the **terms**

Example: $t_n = 2n + 1$
$3, 5, 7, 9, 11$

- When given a **sequence**, it is possible to find the formula

Example: $1, 8, 27, 64, 125$
$t_n = n^3$

- A **series** is the **sum** of a **sequence**

Arithmetic sequences and series

- Difference between consecutive **terms** is a **constant**
- This is called a **arithmetic sequence**
- First **term** t_1 is denoted by a
- Each **term** after the first is found by **adding** a **constant**
- This is called the **common difference** denoted by d of the preceding **term**

Formula: $t_n = a + (n-1)d$

Example: $\{8, 12, 16\}$

$\therefore a = 8, d = 4$

$t_n = a + (n-1)d$

$t_{19} = a + (19-1)d$

$t_{19} = 8 + 18d$

$t_{19} = 8 + 18(4)$

$t_{19} = 8 + 72$

$t_{19} = 80$

or

$t_n = 4n + 4$

$t_{19} = 4(19) + 4$

$t_{19} = 76 + 4$

$t_{19} = 80$

- Applications for **arithmetic sequence**

Example: Find interest earned on $300 over 10 years. The 15th year was $325

$t_{10} = a + (10-1)d$

$300 = a + 9d$

$t_{15} = a + (15-1)d$

$325 = a + 14d$

$a = 325 - 14d$

$300 = 325 - 14d + 9d$

$d = 5$

$a = 255 - 5$

$\frac{250}{5}(100) = 2\%$

- The **sum** of the **terms** in an **arithmetic sequence** is an **arithmetic series**

  Formula: $s_n = \frac{n}{2}(a + t_n)$

- Plug in the values to find the **sum** of the **sequence**

  Example: Find first 5 **terms**

  $\{2, 5, 8, 11, 14\}$

  $s_5 = \dfrac{5}{2}(2 + 14)$

  $s_5 = 40$

Geometric sequences and series

- When you **multiply** the preceding **term** by an **integer**
- The **ratio** of **consecutive terms** is called the **common ratio**
- In **geometric sequences** the first **term** is t_1 denoted by a
- Each **term** after the first is found by **multiplying** the previous term by the **common ratio** r

Example:

$$\{5, -10, 20, -40, 80\}$$
$$t_n = 5(-2)^{n-1}$$
$$t_5 = 5(-2)^{5-1}$$
$$t_5 = 5(-2)^4$$
$$t_5 = 5(16)$$
$$t_5 = 80$$

- General **geometric sequence** is a, ar, ar^2, ar^3 ...
- a is the first **term**, r is the **common ratio**

Formula:

$$t_n = ar^{n-1}, n = \text{natural}, r \neq 0$$
$$\frac{t_2}{t_1} = \frac{ar}{a} = r$$
$$\therefore r = \text{ratio of any successive pair of terms}$$

- Finding the number of **terms**

Example:

$$\{3, 6, 12 \dots 384\}$$
$$t_n = ar^{n-1}$$
$$\frac{384}{3} = \frac{3(2)^{n-1}}{3}$$
$$128 = 2^{n-1}$$
$$2^7 = 2^{n-1}$$
$$7 = n - 1$$
$$8 = n$$

- Finding t_n given 2 **terms**

Example:

$$t_5 = 1875, t_7 = 46875$$
$$\frac{46875}{1875} = \frac{ar^6}{ar^4} \rightarrow r = \pm 5$$

- Applications for **geometric sequence**

Example: Half-life of Iodine is 8 days. Average dose is 12mg. What is the does after 112 days?

$a = 12\text{mg}$

$r = \frac{1}{2}$

$n = 15 \because \frac{112}{8} = 14(+1) \because t_0 = 0$

$\therefore t_{15} = 12\left(\frac{1}{2}\right)^{15-1}$

$t_{15} = 7.3e^{-4}\text{mg}$

$\therefore \text{genreal term} = t_n = 12\left(\frac{1}{2}\right)^{n-1}$

- The **sum** of the **terms** in an **geometric sequence** is an **geometric series**

Formula: $s_n = \frac{a(r^n-1)}{r-1}$

Recursion Formula

- Formulas used to calculate a **term** based on previous **terms**
- **Geometric** and **arithmetic sequences** are **explicit**, meaning they do not use previous **terms**
- When given 2 **terms** and the formula, it is possible to find another **term**

Example: Solve for t_4

$t_1 = -1$
$t_2 = 1$
$t_n = 2t_{n-2} + 4t_{n-1}$
$t_4 = 2t_{4-2} + 4t_{4-1} \rightarrow$ Can't solve because term 3 is not given: $2t_2 + 4t_{\boxed{3}}$
$\therefore t_3 = 2t_{3-2} + 4t_{3-1}$
$t_3 = 2$
$t_4 = 2t_{4-2} + 4t_{4-1}$
$t_4 = 2(1) + 4(2)$
$t_4 = 10$

- When given the **sequence**, find next **term**

Example: $\{1, 2, 4, 7, 11, 16\}$

$t_1 = 1$
$t_2 = 1 + t_1$
$t_3 = 2 + t_2 \ldots$
$\therefore t_n = (n-1) + t_{n-1}$
$t_5 = 4 + t_4$
$t_5 = 4 + 7$
$t_5 = 11$

- When given the **sequence**, get the formula

Example: $\{4, 5, 20, 100, 2000\}$

$t_1 = 4$
$t_2 = 5$
$t_3 = t_1(t_2)$
$t_4 = t_2(t_3)$
$t_5 = t_3(t_4)$
$\therefore t_n = (t_{n-2})(t_{n-1})$

Pascal's Triangle

Pascal's triangle is an **array**. **Pascal's triangle** is useful for **probability** calculations.

- Based on the **sum** of 2 terms immediately above when visually laid out

Example:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \\ 1\ 6\ 15\ 20\ 15\ 6\ 1 \\ \ldots \end{array}$$

- If $t_{n,r}$ represents the **term** in row n, position r

Formula: $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$

$$\begin{array}{c} t_{0,0} \\ t_{1,0}\ t_{1,1} \\ t_{2,0}\ t_{2,1}\ t_{2,2} \\ \ldots \end{array}$$

Example: Given the first 6 **terms** in row 25 of **Pascal's triangle**. Find the first 6 terms in row 26

$\{1, 25, 300, 2300, 12650, 5130\}$

$r = 25: 1, 25, 300, 2300, 12650, 5130 \ldots$

$\therefore r = 26: 1, 26, 325, 2600, 14950, 65780 \ldots$

Binomial Theorem

Recall that a **binomial** is a **polynomial** with 2 **terms**

- General formula for a **binomial**

Formula: $a + b$

- **Expanding** $(a+b)^n$ can be solved through **binomial expansion**
- Using **Pascal's triangle** use n as the row number and **multiply** the **coefficients** through the formula

Example:

Let $a = 2x$
Let $b = -1$
$(2x-1)^4$
$\text{Coefficients} = 1, 4, 6, 4, 1$
$\therefore (2x-1)^4$
$$= 1(2x)^4(-1)^0 + 4(2x)^3(-1)^1 + 6(2x)^2(-1)^2 + 4(2x)^1(-1)^3 + 4(2x)^0(-1)^4$$
$$(2x-1)^4 = 16x^4 + 4(8x^3)(-1) + 6(4x^2)(1) + 4(2x)(-1) + 1$$
$$(2x-1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$$

Financial Math

There are many applications and **algebraic** uses for **financial math**

- **Compound interest** is a **geometric formula**

Formula: $A = P(1+i)^n$
A: accumulated amount at end of term
P: principle (amount)
i: interest represented by $\left(\frac{r}{n}\right)$, r: interest rate per year, n: compounding periods
n: represented by Ny, y: number of years

- **Substitute variable terms**

Example: For $1000 at an interst of 7% per year for 10 years, compounded semi-annually (twice a year)

$\therefore i = \frac{7\%}{2}$
$i = 3.5\%$ or 0.035
$\therefore n = 2(10)$
$n = 20$
$\therefore A = 1000(1 + 0.035)^{20}$
$A = 1989.79$

- Rearrange the formula to solve for different situations

Example: Find the doubling time, compounded semi- annually 8 years

$\therefore P = 1000$
$\therefore A = 2000$
$\therefore i = \frac{r}{2}$
$\therefore 2000 = 1000(1+i)^{16}$
$2000 = 1000\left(1 + \frac{r}{2}\right)^{16}$
$2 = \left(1 + \frac{r}{2}\right)^{16}$
$r = 0.088 \rightarrow 8.8\%$

- Present value used to find the amount needed to achieve a certain amount later

Formula: $P = A(1+i)^{-n}$

Example: Want \$1000000 at the age of 35, presently 18, (18 years difference), an interest of 8%, compounded quarterly (four times a year)

$$\therefore i = \frac{0.08}{4} \rightarrow 0.02$$
$$\therefore n = 18(4) \rightarrow 72$$
$$P = 1000000(1.02)^{-72}$$
$$P = 240318.74$$

- Ordinary annuity is compounding interest with consecutive inputs of value

Formula: $A = \frac{R[(1+i)^n - 1]}{i}$

Example:

$$a = 1500$$
$$n = 5$$
$$i = 12\%$$
$$A = \frac{1500(1.12^5 - 1)}{0.12}$$
$$A = 9529.27$$

- Present value or ordinary annuity reconstructs the ordinary annuity formula

Formula: $P = \frac{R[1-(1+i)^{-n}]}{i}$

Example: $R = 10000$

Twice a year for 5 years compounded semi-annually

$$i = 15\%$$
$$P = \frac{\left(10000(1 - 1.075^{-10})\right)}{0.075}$$
$$P = 68640.81$$

Graphing

Direct and Partial Variation

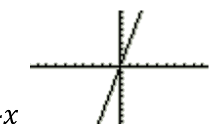
Direct Variation

- A **direct variation** is a relationship between 2 **variables** in which one **variable** is a **constant multiple** of the other

  Examples: $y = 5x, y = x, y = kx, \frac{1}{2}y = x, y = 2x$

- The **constant of variation** is the number before the **variable**

  Example: $y = 3x$, 3 is the constant of variation

  Example: This graph shows an example of **direct variation** $y = 4x$ 

- In **direct variation**, the **line** will always go through the **origin** $(x, y) = (0, 0)$

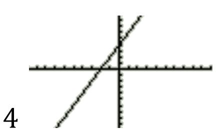
Partial Variation

- A **partial variation** is a relationship between 2 **variables** there is a **constant multiple** and a **constant number**

  Examples: $y = 3x + 5, y = mx + b$

- The **constant of variation** is the number before the **variable** and the **constant number** is the number after the **variable**

  Example: $y = 3x + 5$, 3 is the **constant of variation** and 5 is the **constant number**

  Example: This graph shows an example of **direct variation** $y = 2x + 4$ 

- **Partial variation** never goes through the **origin**
- In both **direct** and **parital varaition**, the **line** must always be stright

Plotting

When **graphing**, you have 2 axis. You have an x axis and an y axis. The x axis is always the horizontal **line** and the y axis is the vertical **line**. These **lines** both interest at the **origin** $(x, y) = (0, 0)$

- A **point** identifies a position and can be represented by numbers (**coordinates**) or a **variable**
- **Coordinates** are **points** on a **graph** which can range to any **integer**
- An **ordered pair** is also a set of **coordinates**
- When given a **coordinate**, the first number is the x -coordinate and the second number is the y -coordinate

Formula: (x, y)

Example: $(2, 5)$

- **Coordinates** can be negative as well

Example: $(-5, 4)$

Example: $(-2, -4)$

Example: $(0, -3)$

Quadrants

- There are 4 **quadrants** when **graphing**. The **quadrants** are labelled in counter clockwise order starting at the top right **quadrant**. **Quadrants** are also known as the **cast**

2	**1**
3	**4**

- When asked of what **quadrant** a **point** is in, refer to the diagram above or to its positive or negative attribute

Given: (x, y)

Qudrant 1: $(+,+)$

Qudrant 2: $(-,+)$

Qudrant 3: $(-,-)$

Qudrant 4: $(+,-)$

- If a **point** lands on an axis (line) or origin, it has no **quadrant**
- Ensure that you label the axis and **points**

Slope

Slope is simply put is the **rate of change**; the **slope** is expressed as a **fraction**. The variable term of slope is m

Example: $m = \frac{3}{4}$

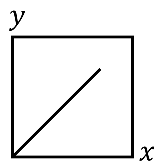
- A plane is a surface that goes on forever
- The Cartesian plane is a grid (**graph**) that has x and y **coordinates**
- The simplest way to find **slope** when given 2 **points** on a **graph** is to measure rise over run

Formula: $m = \frac{\text{rise}}{\text{run}}$

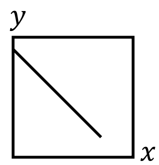
Example: $m = \frac{5}{10} = \frac{1}{2}$

- **Slope** can be referred to the flowing terms: **slope**, angle, steepness, grade, incline or rate of change

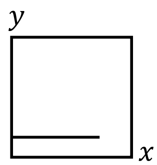
Slopes of line segments



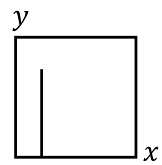
Rises to the right
m = positive



Falls to the right
m = negative



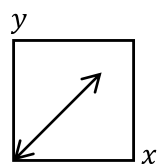
Horizontal line
$y = b$
m = 0



Vertical Line
m = undefined

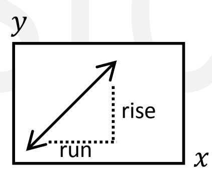
- Although the above examples show **line segments**, these **segments** are actually **lines**. The **notation** for this should be that the **lines** end with arrows indicating that it goes on forever

Example:



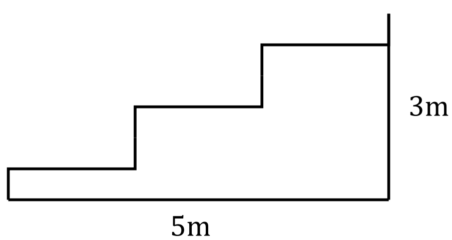
- Rise is the vertical distance between 2 **pointes** (up and down); change in y or Δy
- Run is the horizontal distance between 2 **points** (across); change in x or Δx

Example:



- A **fractional slope** is NEVER expressed in any units

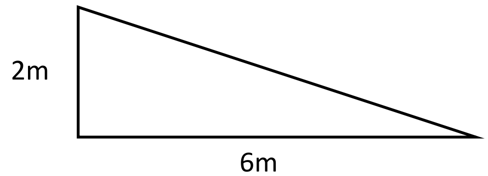
Example:



$$m = \frac{\text{rise}}{\text{run}} = \frac{3}{5}$$

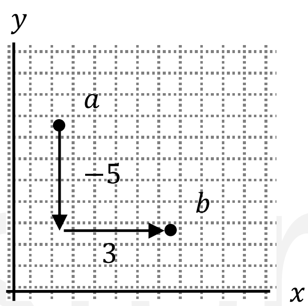
- On a grid, you can physically count the rise and run
- On a grid it is critical that you ensure you start your **slope** from the point furthest to the left, then calculate the run, NEVER the other way around
- The run should never be negative
- You can only get a negative **slope** by encountering a negative rise. This may only happen if the **point** furthest to the left is also higher than the alternative **point**; this is also a **slope** that is falling to the right or a downward trend, all positive **slopes** are upward trends

Example:



$m = \frac{-2}{6} = -\frac{1}{3}$

Example:



$m = \frac{-5}{3} = -\frac{5}{3}$

- In some cases, you will be given a **slope** and a **point**, to find the other **point**, simply break down the **slope** into rise and run and add the rise to the y -coordinate and run to the x –coordinate

Example: Point $A(2,1)$, $m = \frac{5}{2}$

$B = A(2,1) + m$

$B = \frac{1}{2} + \frac{5}{2}$

$B = \frac{4}{6}$

Slope Formula

When given 2 points, you can find the **slope** by both **plotting** the **points** and reading the **graph**, or use **slope** formula

- **Slope** formula is calculated by taking the **coordinates** of 2 **points** and **subtracting** their y and x values

  Formula in professional: $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{\Delta y}{\Delta x}$ (change in y values over change in x values)

  Formula in linear: $m = (y\_2 - y\_1\ )/(x\_2 - x\_1\ )$ or $\Delta y/\Delta x$

- From 2 **points**, you take a y –coordinate and subtract it with the alternative y –coordinate and **divide** it by the difference between the x –coordinates. It is irrelevant of which x or y – **coordinate** is subtracted as long as you remain consistent

  Example: $A(2,5)\ B(11,25) m = \frac{y_2 - y_1}{x_2 - x_1}$

  $$m = \frac{25 - 5}{11 - 2}$$
  $$m = \frac{20}{9}$$

- Ensure that when you are dealing with negative **coordinates**, treat it as a negative **integer** and **bracket** the **coordinate**

  Example: $S(-4,-6)\ T(5,-3)$

  $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
  $$m = \frac{-3 - (-6)}{5 - (-4)}$$
  $$m = \frac{-3 + 6}{5 + 4}$$
  $$m = \frac{3}{9} = \frac{1}{3}$$

Rate of Change

The **rate of change** is a change in one quantity relative to the change in another quantity

- **Rate of change** requires units. Units are given from the x or y labels
- **Rate of change** is similar to calculating **slope**
- Unit should be **expressed** as y over x

Example: km/h

- To calculate the **rate of change**, simply calculate the $\frac{\Delta y}{\Delta x}$

Example: change in distance over change in time

Example: $\frac{5}{20} = \frac{1}{4}$

- Always **express rate of change** as a **decimal**

Example: 0.25km/h

- If the **rate of change** is positive, the **slope** is ascending and if it is negative, the **slope** is descending

Identifying linear and Non-linear relations

There are 3 ways of identifying if a relationship is **linear** of **non-linear**

1. Graph — If the **line** is straight, it is a **linear** relationship
2. Equation — If x is raised to the **exponent** 1, it is a **linear** relationship
3. Table — If the **first differences** are equal, it is a **linear** relationship

- **First differences** are calculated by using a table. The x axis is the first column, and should range to any **integer** and start from any **integer**. The y axis is the second column; it is labelled as either y values or the formula of the line in y intercept form. The third column is the first difference column
- **First difference** is also known as **finite differences**
- How first differences are calculated is by **subtracting** a y value by the previous y value
- Plug in the x values into the y value
- If ALL the first differences are equal, then the relationship is **linear**, otherwise, the relationship is **non-linear**

Formula:

x values	y values	First Difference

- The first value in the series can't have a **first difference** for it has no previous value
- Sometimes, the y values column will already be set, otherwise, if only the formula is given, then plug in the x values

Example:

Side length (cm)	Volume (cm^3)	First Difference
$\mathbf{1}$	$\mathbf{1}$	-
$\mathbf{2}$	$\mathbf{8}$	$\mathbf{1-8=7}$
$\mathbf{3}$	$\mathbf{27}$	$\mathbf{27-8=19}$
$\mathbf{4}$	$\mathbf{64}$	$\mathbf{64-27=37}$
$\mathbf{5}$	$\mathbf{125}$	$\mathbf{125-64=61}$

$\because 7 \neq 19$ $\therefore$ This relationship is not linear

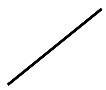
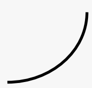
- It is not necessary to continue the first differences if even 1 of the relations are not equal

Example:

x	$3x+2=9$	First Difference
$\mathbf{0}$	$\mathbf{3(0)+2=2}$	-
$\mathbf{1}$	$\mathbf{3(1)+2=5}$	$\mathbf{3}$
$\mathbf{2}$	$\mathbf{3(2)+2=8}$	$\mathbf{3}$
$\mathbf{3}$	$\mathbf{3(3)+2=11}$	$\mathbf{3}$
$\mathbf{4}$	$\mathbf{3(4)+2=14}$	$\mathbf{3}$

∵ The **first differences** are **constant**, this is a **linear** relationship

The chart below displays the 3 techniques of how to identify **linear** and **non-linear** relationships

	Graph		**Equation**		**Table**	
	Identify	Example	Identify	Example	Identify	Example
Linear	Straight Line		x to the power of 1	x^1	Constant	3,3,3
Non-Linear	Not a straight line		x to the power other than 1	x^3	Not constant	$-1,4,5$

Collinear

If 3 or more **points** have an equal **slope**, then the **points** are **collinear** (on the same line) else if even 1 **point** doesn't have a similar **slope** to the other **points**, then the **points** are not **collinear** (not on the same line)

- When given 3 or more **points**, find the **slope** by combining 2 **points** and using **slope** formula. Pair up all **points** and until there is a difference of **slope**, do not discontinue, else the **points** are **collinear**
- Ensure that the **slope** is in lowest form before making any judgements

Example: Given the points $A(-1,-1)$ $B(2,1)$ $C(5,3)$

$$mAB = \frac{y_2-y_1}{x_2-x_1} = \frac{1-(-1)}{2-(-1)} = \frac{2}{3}$$

$$mAC = \frac{y_2-y_1}{x_2-x_1} = \frac{3-(-1)}{5-(-1)} = \frac{4}{6} = \frac{2}{3}$$

$$mBC = \frac{y_2-y_1}{x_2-x_1} = \frac{3-1}{5-2} = \frac{2}{3}$$

$\because mAB = mAC = mBC \;\therefore A, B, C$ are **collinear**

Example: Given the points $D(1,2)$ $E(5,6)$ $F(9,9)$

$$mDE = \frac{y_2-y_1}{x_2-x_1} = \frac{6-2}{5-1} = \frac{4}{4} = 1$$

$$mDF = \frac{y_2-y_1}{x_2-x_1} = \frac{9-2}{9-1} = \frac{7}{8}$$

$\because mDE \neq mDF$ they are not **collinear**

Graphing Equations

A **line** can be **graphed** using a **table of values**

- Remember that when given a **table of values**, the **plotting** does not make a **line**, but a **line segment**. Therefore, do not draw arrows at the end of the **line** when **graphed**

Example: $y = x$

x	y	(x, y)
-2	-2	$(-2, -2)$
-1	-1	$(-1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$

These 5 **points** plotted would go through the **origin** in an upward trend

- Ensure that you always label the axis and write the **equation** of the **line** in slope y-intercept form

Example: $y = -2x + 3$

x	$-2x + 3 = y$	(x, y)
-2	$-2(-2) + 3 = 7$	$(-2, 7)$
-1	$-2(-1) + 3 = 5$	$(-1, 5)$
0	$-2(0) + 3 = 3$	$(0, 3)$
1	$-2(1) + 3 = 1$	$(1, 1)$
2	$-2(2) + 3 = -1$	$(2, -1)$

Label the **line** as $y = -2x + 3$

- Always be sure that the points satisfy the **equatioon**

Equation of a line

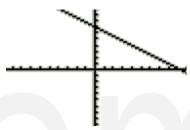
The **equation** $y = mx + b$ is the general form of a **line**. This **term** is also referred to as y-intercept form or **slope** y-intercept form

- y represents the y axis
- m represents the **slope** (formula or rise over run)
- b represents y-intercept (y-int) which is where the **line** intercepts with the y axis

Example: $y = \frac{1}{2}x + 5$

$$m = \frac{1}{2}$$
$$b = 5$$

- To **graph** this, you start at the y-intercept point or b; $(0, b)$
- Use the **slope** to guide you to the net **point** by using rise over run (remember if negative, the **slope** is going down)
- Ensure to label axis's and the line and remember to put arrows for it is not a **line segment**

Example: $y = -\frac{4}{3}x + 7$ 

- When given the **slope** and y-intercept, you can create a formula

Example: $m = \frac{3}{4}; b = -2$

$$\therefore y = \frac{3}{4}x - 2$$

- x-intercept (x-int) is similar to y-intercept except that x-intercept is where the **line** intercepts with the x-axis
- When only given $y =$ or $x =$, it is either a vertical **line** or horizontal **line**. The **integer** given is where the **intercept** is

Example: $y = 3$; Horizontal line

Example: $x = -5$; Vertical line

- When something is missing from the **equation**, it indicates that the value is 0

Example: $y = mx$; goes through the **origin**/**direct variation** for it has no b value

Standard Form

Standard form has all **variables** and **constants** of $y = mx + b$ but is **expressed** as $Ax + By + C = 0$. There are several conditions that must be true for an **equation** to be in standard form

- A, B, C are all **integers**
- A, B, C can be 0 but A and B can't both be 0 at the same time
- Always write the x-term, then y-term then **integer term**
- All the **terms** must be written on the left side
- Always write it in lowest terms
- x can't be negative when in **standard form**, simply **multiply** each term by -1

Example: $3x + 2y + 1 = 0$

- To convert standard form into **slope** y-intercept form, isolate the y term

Example:
$$3x + \boxed{2y} + 1 = 0$$
$$2y = -3x - 1$$
$$\frac{2y}{2} = -\frac{3}{2}x - \frac{1}{2}$$
$$y = -\frac{3}{2}x - \frac{1}{2}$$

- To convert **slope** y-intercept form into **standard form**, move all terms to the right side, in $x, y,$ # order

Example:
$$y = 3x + 5$$
$$-3x + y = 5$$
$$-3x + y - 5 = 0$$
$$3x - y + 5 = 0$$

Intercepts

In a **linear line** or **line segment**, there can only be up to 2 intercepts, one x and y. To find the intercepts in an **equation** you must first convert the **equation** into **standard form** if not already

Example: $y = -2x + 4$
$2x + y = 4$
$2x + y - 4 = 0$

- To find the x-intercept, in the **standard form equation**, make $y = 0$, then solve

Example: $2x + y - 4 = 0$
$2x + 0 - 4 = 0$
$2x - 4 = 0$
$2x = 4$
$\frac{2x}{2} = \frac{4}{2}$
$x = 2$

- To find the y-intercept, in the **standard form equation**, make $x = 0$, then solve

Example: $2x + y - 4 = 0$
$2(0) + y - 4 = 0$
$y - 4 = 0$
$y = 4$

- To **graph** this, simply convert each intercept into a **coordinate**, then plot the **coordinates** and link them to form a **line segment**

Example: $x = 2;\ \therefore A(2, 0)$

Example: $y = 4;\ \therefore B(0, 4)$

- When given intercepts, and you are asked to find the **slope**, first convert each intercept into **coordinates**

Example: $x = 2;\ \therefore A(2, 0);\ y = 4;\ \therefore B(0, 4)$

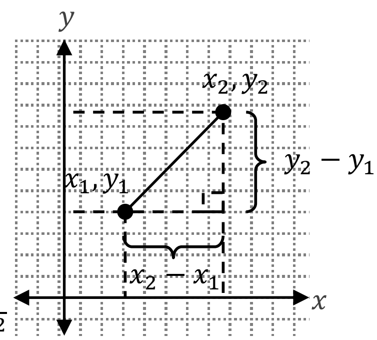
- When given 2 **coordinates**, you can calculate **slope** using the **slope** formula

Example: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 2} = \frac{4}{-2} = -2$
$\therefore m = -2$

Length of a Line Segment

On a **grid**, finding the length of a **line segment** is similar to the **equation of a line**.

- To solve, simply plug in the **coordinates** given



Formula: $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example: Find the distance between $A(2,7), B(-9,4)$

$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$L = \sqrt{(-9-2)^2 + (4-7)^2}$

$L = \sqrt{121 + 9}$

$L = \sqrt{130 \text{ units}}$

- When given a vertical or horizontal **line** and a coordinate, simply take the missing **coordinate** from the **point** and plug it into the **line equation**

Example: $y = 3, (-2, -2)$

$\therefore (-2,3)$

$x = -2$

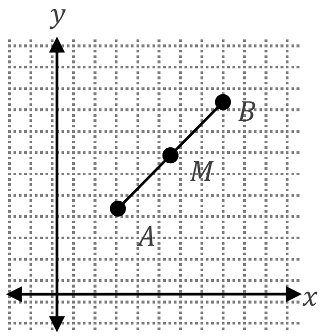
$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$L = \sqrt{(-2+2)^2 + (3+2)^2}$

$L = \sqrt{25}$

$L = 5$

- The **midpoint** of a **line segment** is the **average** of its **endpoints**



Formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example: Find the midpoint of $(2,0), (0,-3)$

$=\left(\frac{2+0}{2}, \frac{0-3}{2}\right)$

$=\left(1, -\frac{3}{2}\right)$

Example: Find the midpoint of $(a, 2b), (2a, 3b)$

$=\left(\frac{a+2a}{2}, \frac{2a+3b}{2}\right)$

$=\left(\frac{3a}{1}, \frac{5b}{2}\right)$

Example: Let $P(6,-8)$ and $Q(6, R)$. Midpoint of PQ is $(t, 4)$. Find t and R

$t=\frac{6+6}{2}$

$=\frac{12}{2}$

$=6$

$4=\frac{-8+R}{2}$

$8=-8+R$

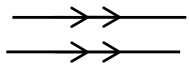
$R=16$

Parallel and Perpendicular Lines

When given 2 **line equations** in slope y-intercept form, there are ways to determine whether they are **parallel** or **perpendicular**

- **Parallel lines** never cross and remain the same distance apart; Symbol: ∥
- **Parallel lines** are marked with arrows pointing in the same direction

Example:



- **Parallel lines** can be easily identified when in **slope** y-intercept form for the **slopes** of the 2 lines will be equivalent

Example: Which lines are **parallel** when given the **coordinates**?

$$mAB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-2}{3-1} = \frac{5}{4}$$

$$mCD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-1}{7-3} = \frac{5}{4}$$

$$mEF = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-1}{3-(-4)} = -\frac{3}{7}$$

$$mGH = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-(-3)}{3-(-4)} = -\frac{3}{7}$$

$$\because mAB = mCD \ \therefore AB \parallel CD; \ \because mEF = mGH \ \therefore EF \parallel GH$$

- **Perpendicular lines** cross at a $90°$ angle; Symbol: ⊥
- **Perpendicular lines** are marked with a square where the intersection is

Example:

- **Perpendicular lines** can be easily identified when in **slope** y-intercept form for one of the **slopes** of the 2 **lines** will be a negative **reciprocal** of the other

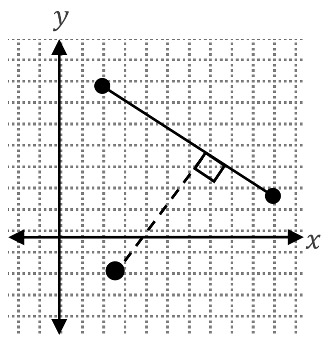
Example: Are these lines **perpendicular** given the **slopes**?

$mAB = -\frac{1}{8}$
$mCD = 8 \because mAB \perp mCD \;\therefore AB \perp CD$

Distance from a point to a line

- The distance from a **point** to a **line** is always considered to be **perpendicular** distance (shortest distance)

Example:



- To solve, you will be given 2 things, both a point and a **lines**
- Find the **perpendicular slope** from the **line** and make an **equation** from the **coordinates** and the **slope**
- Next, find the **point of intersection** and then find the distance from the **point** and the **intersection**

Example: Find the distance from $(-3,1)$ to $y = x + 10$

$y = x + 10$

$m \perp = -1$

$y - y_1 = m(x - x_1)$

$y = 1 = -1(x + 3)$

$y - 1 = -x - 3$

$y = -x - 2y = y$

$-x - 2 = x + 10$

$2x = -12$

$x = -6$

$y = x + 10$

$y = -6 + 10$

$y = 4$

$\therefore$ the POI $= (-6,4) L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$L = \sqrt{(-6 + 3)^2 + (4 - 1)^2}$

$L = \sqrt{(-3)^2 + (3)^2}$

$L = \sqrt{9 + 9}$

$L = \sqrt{18}$

$L = \sqrt{18}$ units

Example: Find the distance from $(4,6)$ to the line $x = -4$

distance $= 4 - (-4)$

$= 8$ units
$y = 8$

Example: $y = x - 4, (0,0)$
$m \perp = -1$
$y - 0 = -1(x - 0)$
$y = -xy = y$
$-x = x - 4$
$x = 2$
$y = 2 - 4$
$y = -2$
$(2, -2)$
$L = \sqrt{(2 - 0)^2 + (-2 - 0)^2}$
$L = \sqrt{4 + 4}$
$L = \sqrt{8}$

Finding Equations

These are several ways to find an **equation** of a **line** when given enough information. From the information given, you plug in items into the **equation** and solve

- The simplest equation we can use to allow easy **substitution** is represented through this formula

Formula: $y - y_1 = m(x - x_1)$

Solve for b when given a **slope** and **point**

Example: $A(x, y) = A(2,2)$ and $m = 5 = \frac{5}{1}$

- Plug in the information given into the representative **variables** and solve for the missing **variable**. Use the **coordinates** of the **point** as your x and y **variables** in your **equation**

Example: Without $y - y_1 = m(x - x_1)$

$y = mx + b$
$2 = 5(2) + b$
$2 - 2 = 10 + b - 2$
$0 - b = 10 + b - 2 - b$
$-b = 10 - 2$
$-b = 10 - 2$
$b = -10 + 2$
$b = -8b = -8$
$\therefore y = 5x - 8$

With $y - y_1 = m(x - x_1)$

$y - y_1 = m(x - x_1)$
$y - 2 = 5(x - 2)$
$y - 2 = 5x - 10$
$y = 5x - 8$

- Plug in missing value into the final **equation**

Example: $y = 5x - 8$

Solve for b when given an **equation** and a **point**

- Any **point** given on the **line** must satisfy the **equation** for the **line**

Example: $y = -2x + b$ and $A(2,1)$
to solve b, substitute the **point** into the **equation**
$y = -x + b$
$1 = -2(2) + b$
$1 = -4 + b$
$1 + 4 = -4 + b + 5$
$b = 5$
$\therefore y = -2x + 5$

Solve for m when given b and a **point**

- We require 2 **points** in order to find **slope** by using the **slope equation**

Example: $y = mx - 2$ and $A(5,4)$
to solve m, use b as a point (intercept)
$A(5,4)$ and B(0,-2) (y-intercept)
$m = \frac{y_2 - y_1}{x_2 - x_1}$
$m = \frac{-2 - 4}{0 - 5}$
$m = -\frac{6}{5}$
$\therefore y = -\frac{6}{5}x - 2$

- We given 2 **points** solve m and then **substitute** it into $y - y_1 = m(x - x_1)$ to solve

Example: Find the equation of a line through $A(-7,2), B(6,-9)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{-9 - 2}{-7 - 6}$$
$$m = \frac{11}{13} y - y_1 = m(x - x_1)$$
$$y - 2 = -\frac{11}{13}(x + 7)$$
$$y = -\frac{11}{13}x - \frac{77}{13} + 2$$
$$y = -\frac{11}{13}x - \frac{77}{13} + \frac{26}{13}$$
$$y = -\frac{11}{13}x - \frac{51}{13}$$

Making Equations

When given 2 **points**, you can form an **equation** of a **line**

- From the 2 **points**, find the **slope** through **slope equation**

Example: $A(-1,3)$ and $B(1,-1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{-1 - 3}{1 - (-1)}$$
$$m = -\frac{4}{2} = -2$$

- Then substitute in either of the **points** into $y = mx + b$ to find b

Example: $A(-1,3)$ and $B(1,-1)$ and $m = -2$

$$y = mx + b$$
$$3 = -2(-1) + b$$
$$3 = 2 + b$$
$$b = 1$$
$$\therefore y = -2x + 1$$

- It is irrelevant of which **point** you substitute in, be sure to watch where the x and y **coordinates** go

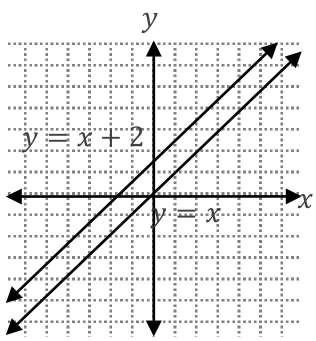
Example: $A(-3,-2)$ and $B(6,-8)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\mathrm{m} = \frac{-8—(-2)}{6—(-3)}$$
$$m = -\frac{6}{9} = -\frac{2}{3}$$
$$\mathrm{y = mx + b}$$
$$-2 = -\frac{2}{3}(3) + \mathrm{b}$$
$$-2 = 2 + \mathrm{b}$$
$$-2 - 2 = 2 + \mathrm{b} - 2$$
$$\mathrm{b} = -4$$
$$\therefore \mathrm{y} = -\frac{2}{3\mathrm{x}} - 4$$

Linear Systems

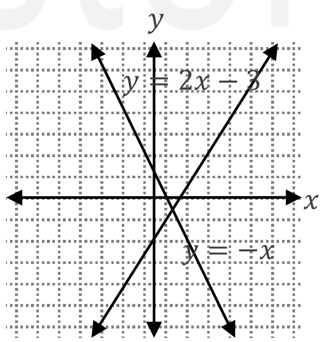
A **linear system** is a solution for 2 **equations/lines** where a **point** or set of **points** satisfies both **equations/lines**. There are 3 types of **solutions**

- **Point of intersection** (**POI**) is the point(s) that satisfies the **equations/lines**
- 2 **parallel lines**
- Never cross
- No point that satisfies both **equations/lines**. Also referred to as **coincident** lines

Example: 

no POI

- 2 non-**parallel lines**
- Cross once/intercept once
- Only 1 **point** that satisfies both **equations/lines**

Example: 

POI $= (1, -1)$

- 2 identical **lines**
- Every **point** intercepts
- Every **point** satisfies the **equations/lines**

Example: $2x + 3y = 6$ and $4x + 6y = 12$

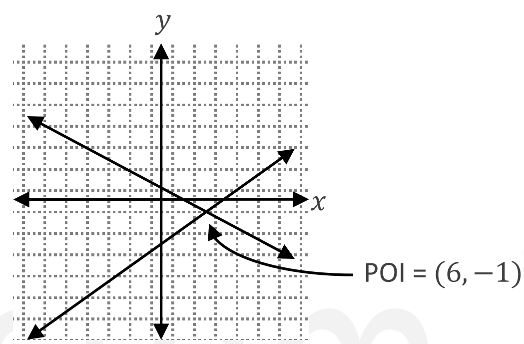
Finding the **point of intersection** (**POI**) can be found through **graphing equations/lines**. The **point of intersection** is where the 2 **lines** cross/intercept. Therefore, this point satisfies both **equations/lines** thus giving us the **solution** to the **linear system**

- When using the **graphing** method, it's best to have both **equations/lines** in **slope** y-intercept form. You can also use x and y-intercepts

Formula: $y = y$

Example: $y = \frac{2}{3} - 5$ and $x + 2y = 4$

$x + 2y = 4; x\text{-int} = (4{,}0), y\text{-int} = (0{,}2)$



- To verify that the **POI** is correct, simply plug in the **POI** into both **equations**. If the left side of the **equation** is equal to the right side, the **POI** is correct

Example: $y = \frac{2}{3}x - 5$

$$-1 = \frac{2}{3}(6) - 5$$

$-1 = 4 - 5$

$-1 = -1$

LS (Left side) = RS (Right side)

$x + 2y = 4$

$6 + 2(-1) = 4$

$6 - 2 = 4$

$4 = 4$

LS (Left side) = RS (Right side)

LS = RS $\therefore$ the point $(6, -1)$ satisfies both **equations/lines**

$\because$ the **point** is on both **lines** it is the **solution** to the **linear system**

Finding the point of intersection (POI) can be found algebraically

- To solve algebraically, the first step is to make each **equation** equal to each other (remove y)
- Ensure both **equations** are in **slope** y-intercept, else convert it into **slope** y-intercept
- Then solve (isolate x)

Example:

$$y = 30x + 50 \text{ and } c = 35x + 40$$
$$30x + 50 = 35x + 40$$
$$30x + 50 - 50 = 35x + 40 - 50$$
$$30x = 35x - 10$$
$$30x - 35x = 35x - 10 - 35x$$
$$\frac{-5x}{-5} = \frac{-10}{-5}$$
$$x = 2$$

- Use the result of this and take as the x **coordinate** of the **POI**
- Next, to solve the y coordinate, plug the x coordinate from the **POI** into either **equation**

Example:

$$y = 30x + 50$$
$$y = 30(2) + 50$$
$$y = 60 + 50$$
$$y = 110$$
$$\therefore \text{ the } \textbf{Point of intersection} \text{ or } \textbf{POI} = (2,110)$$

Finding the point of intersection (POI) can be found through substitution

- First you must convert one of the **equations** to solve for x(isolate)
- Substitute the result of x into the alternate **equation** and solve for y
- Plug y into the x statement to solve for x
- x and y can be switched for the above statement

Example: $5x + y = 11; x - y = -7$

$x = y - 75(y - 7) + y = 11$

$5y - 35 + y = 11$

$6y = 11 + 35$

$6y = 46$

$y = \frac{46}{6} = \frac{23}{3} x = \frac{23}{3} - 7$

$x = \frac{23}{3} - \frac{21}{3}$

$x = \frac{2}{3} \therefore \text{ the POI} = \left(\frac{2}{3}, \frac{23}{3}\right)$

Finding the point of intersection (POI) can be found through elimination

- Simplify the **equations**, eliminate any **fractions** and **collect like terms**
- First you must convert both of the **equations** to align with similar **variables**
- Add the 2 **equations** in aligned order and eliminate similar **variables** and **constantans**
- Isolate 1 of the **variables** and then substitute it into one of the **equations**

Example: $4x - 3y = -10; 3y + 2x = 32$

$4x - 3y = -10$

$\underline{2x + 3y = 32}$

$6x = 12$

$x = 2$

$4x - 3y = -10$

$4(2) - 3y = -10$

$8 - 3y = -10$

$-3y = -18$

$y = 6$

$\therefore$ the POI $= (2,6)$

Example: $6x - 5y = -3 \rightarrow 6x - 5y = -3 \rightarrow \times 2 \rightarrow 12x - 10y = -6$

$2y - 9x = -1 \rightarrow -9x + 2y = -1 \rightarrow \times 5 \rightarrow -45x + 10y = -5$

$12x - 10y = -6$

$\underline{-45x + 10y = -5}$

$-33x = -11$

$x = \frac{1}{3}$

$12x - 10y = -6$

$\frac{12}{1}\left(\frac{1}{3}\right) - 10y = -6$

$\frac{12}{3} - 10y = -6$

$4 - 10y = -6$

$-10y = -10$

$y = 1$

$\therefore$ the POI $= \left(\frac{1}{3}, 1\right)$

- **Elimination** and **fractions**. Remove the fractions by **multiplying** each **term** by a common **denominator** then simply work through

Example:

$\frac{x}{3} - \frac{y}{6} = -\frac{2}{3} \rightarrow \times 6 \rightarrow 6\left(\frac{x}{3}\right) - 6\left(-\frac{y}{6}\right) = 6\left(-\frac{2}{3}\right) \rightarrow 2x - y = -4$

$\frac{x}{12} - \frac{y}{4} = \frac{3}{2} \rightarrow \times 12 \rightarrow 12\left(\frac{x}{12}\right) + 12\left(-\frac{y}{4}\right) = 12\left(\frac{3}{2}\right) \rightarrow x - 3y = 18$

$2x - y = -4 \rightarrow \times 1 \rightarrow 2x - y = -4$

$\underline{x - 3y = 18 \rightarrow \times 2 \rightarrow 2x - 6y = 36}$

$5y = -40$

$y = -8$

$2x(-8) = -4$

$2x = -12$

$x = 6$

$\therefore$ the POI $= (6, -8)$

- **Elimination** and **decimals**. **Multiply** each **term** by 10 to rid of any **decimals**

Example:

$0.5x - 1.3y = 1.23$

$\underline{4x - 2y = 0.6}$

$5x - 13y = 12.3$

$\underline{40x - 20y = 6}$

$200x - 520y = 492$

$\underline{200x - 100y = 30}$

$-420y = 462$

$y = -1.1$

$5x + 14.3 = 12.3$

$5x = -2$

$x = -0.4$

$\therefore$ the POI $= (-0.4, -1.1)$

- If there is no system (**parallel lines**) then there will be no variable to represent the **POI**. The variables should eliminate themselves

Example: $18r + 12s = 30$
$\underline{18r + 12s = 14}$
$undefined = 16$
$\therefore$ the system is parallel

- If the **lines** are equivalent and intercept at every segment(**coincident lines**) then the resolution to both **equations** will be equal

Example: $4x - 3y = 5 \rightarrow \times 2 \rightarrow 8x - 6y = 10$
$\underline{8x - 6y = 10}$
$8x - 6y = 10$
$\underline{8x - 6y = 10}$
$0 = 0$
$\therefore$ the system is ∞ equal

Solving problems using linear systems

- Through either **substitution** or **elimination**, solving word problems can be made simple

Example: Calculate the interest earned on $4000 for 2 years at 6.5% simply

$I = prtp = 4000, r = 0.065, t = 2 = 4000 \times 0.065 \times 2$
$= 520$
$\therefore$ the interest earned was 520

Example: Say you drove 470km in 5 hours from Snowball corners to North Bay. For part of the trip you drove at 90km per hour and part at 100km per hour. How far at each speeds

$v = \frac{d}{t}$
Let x be distance at 100km per hour
Let y be distance at 90km per hour

Distance	Speed	Time
$\boldsymbol{x}$	100	$\frac{x}{100}$
$\boldsymbol{y}$	90	$\frac{y}{90}$

$x + y = 470 \rightarrow \times 900 \frac{x}{100} + \frac{y}{90} = 5 \rightarrow \times 900$
$9x + 9y = 4230$
$\underline{9x + 10y = 4500}$
$-y = -270$
$y = 270$
$x = 470 - 270$
$x = 200$
$\therefore$ 200km at 100km/h and 270km at 90km/h

Example: A small sailboat takes 3 hours to travel 30km with the current and 4 hours to return against the current. Find the speed of the boat and the current

Let $b=$ speed of boat (no current)

Let $c=$ speed of current (no boat)

$d=vt$

	Distance	Velocity	Time
With current	30	$b+c$	3
Against current	30	$b-c$	4

$30=(b+c)\times 3\rightarrow 30=3b+3c$

$\underline{30=(b-c)\times 4\rightarrow 30=4b-4c}$

$12b+12c=120$

$\underline{12b-12c=90}$

$24c=30$

$c=\frac{30}{24}\rightarrow\frac{5}{4}$

$30=4b-4\left(\frac{5}{4}\right)$

$30=4b-5$

$35=4b$

$\frac{35}{4}=b$

$\therefore$ Speed of boat is $\frac{35}{4}$ km/h and current is $\frac{5}{4}$ km/h

Equivalent Equations

- An **equation** can have an infinite number of equivalent forms

Example: $x - 3 = 1$
Multiplied by 2: $2\text{x} - 6 = 2$
Both the equations solve x for 4 even though they are different equations

Equivalent Systems

- Like **equivalent equations** there are **equivalent systems**. Based on the same principle, 2 systems can be alike

Example: System A
$x - y = 3; x + y = 7$
$y = x - 3; y = -x + 7$
$x - 3 = -x + 7$
$2x = 10$
$x = 5$
$y = 5 - 3$
$y = 2$
$\therefore \text{ the POI } = (5,2)$
System B
$x = 5; y = 2$
$\therefore \text{ the POI } = (5,2)$

Types of Graphs

Graphs should generally have the following depending on the type of **graph**

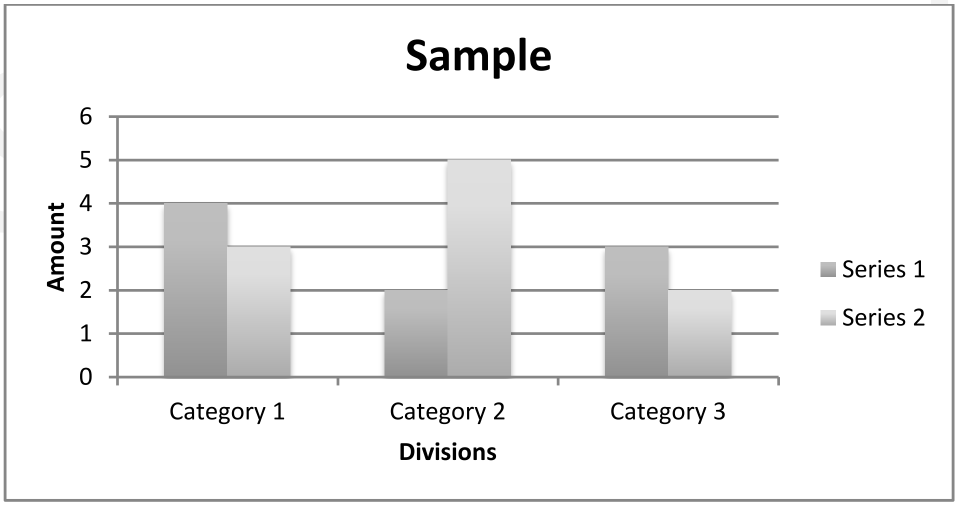
- Chart title
- Axis titles
- Legend/key
- Data labels
- Data table
- Axes
- Gridlines

There are several types of **graphs**; each with its own intended purpose to display results

Bar Graphs

- To show how something changes over time and for comparing
- **Independent variable** is on x axis and **dependant** is on y axis
- Typically used to convey information over a long period of time

Example:

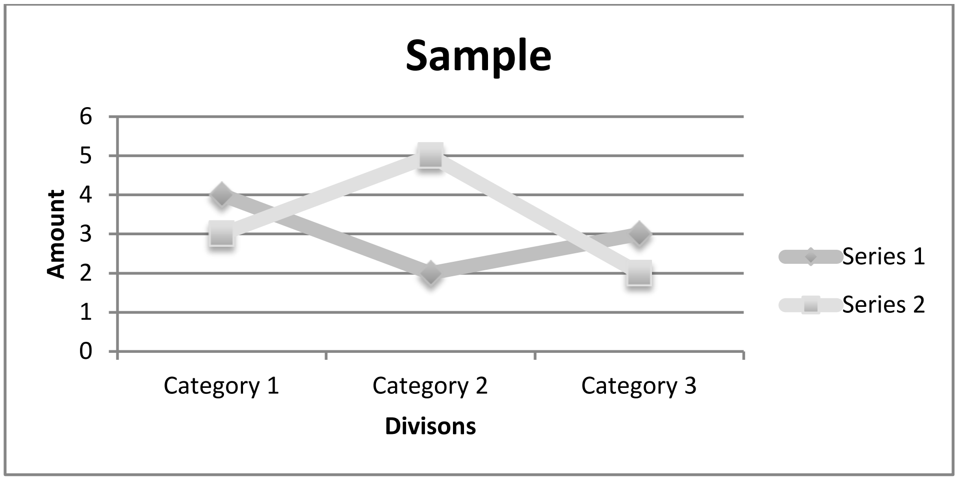


✓Chart title, axis titles, legend, axes, gridlines

Line Graphs

- Comparing 2 **variables**
- **Independent variable** is on x axis and **dependant** is on y axis
- Show trends therefore predictions can be made (patterns)

Example:



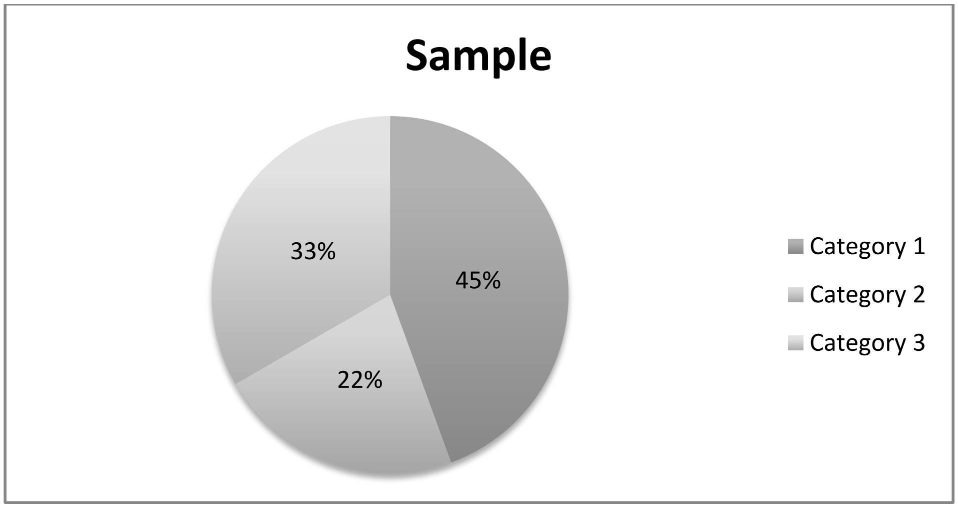
✓Chart title, axis titles, legend, axes, gridlines

Pie Graphs

- Show **percentages** of a whole
- **Dividing** a circle into different section to represent the **percent** of that **variable**
- Finding **percentages** using degrees

Formula: $\frac{\text{total \# in category}}{\text{total \# in sample}} \times 360°$

Example:

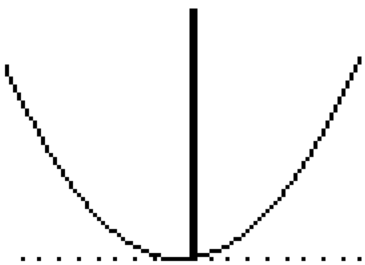


✓Chart title, legend, data labels

Quadratic Functions

A **quadratic function** is a **polynomial**, when **graphed** with **exponents** which will result in a **non-linear** display

- **Expressions** of the form $y = x^2$ are called **quadratic functions**. **Quadratic** means **square**. These **expressions** are also known as a **parabola**



Example: $= 25x^2 + 30x + 9$

Example: $y = x^2$

- **Parabolas** are **symmetric**. They are a **reflection** of each of them along a **line**. In the case of $y = x^2$, the **axis** of **symmetry** is the **line** $x = 0$ ($y - axis$)
- **Parabolas** also have a **vertex**, or turning points. The **vertex** of $y = x^2$ is $(0,0)$. The **vertex** always **intersects** the **axis** of **symmetry**

Example: $x^2 =$ **Quadratic**

- A **relation** is a set of **ordered pairs**

Example: $t\{(6,7),(8,9),(10,11)\}$

- A **Function** is a special relation between **ordered pairs** in which, for every value of x, there is only 1 value of y

Example: $\{(2,3),(4,5),(6,7),(8,9)\}$
The **relation** is a **function**

Example: $\{(6,2),(6,4),(8,6),(10,8)\}$
The **relation** is not a **function**

Example: $\{(-4,8),(-2,4),(0,0),(2,4),(4,8)\}$
The **relation** is a **function**

- The set of first **elements** in a **relation** is called the **domain** of the **relation**. x values are the **domain**

Example: $\{(2,3),(4,5),(6,7),(8,9)\}$
Domain: $\{2,4,6,8\}$

- The set of second **elements** in a relation is called the **range** of the **relation**. y values are the **range**

Example: $\{(2,3),(4,5),(6,7),(8,9)\}$
Range: $\{3,5,7,9\}$

- A **function** can be justified as a set of **ordered pairs** in which, for each **element** in the **domain**, there is exactly one **element** in the **range**

Example: $t\{(6,7),(8,9),(10,11)\}$
The **relation** is a **function**

Example: $q\{(4,2),(4,3),(4,4),(6,9)\}$
The **relation** is not a **function** because there are $3y$ values for $x = 4$

Example: $r\{(-1,-1),(-1,-1),(-2,4),(-1,2),(0,0),(1,2),(2,4)\}$
The **relation** is not a **function** because of $(-1,-1)$ and $(-1,2)$

- The **minimum** value of the **domain** and **range** can be determined by the lowest value within the series
- The **maximum** value of the **domain** and **range** can be determined by the greatest value within the series. Be aware that these values can be infinite or a set of **real numbers**
- A **quadratic function** can have a y intercept

Example: $y = x^2 + 2; y\text{-int} = 2$
Table of Values:

x	y
-4	18
-2	6
0	2
2	6
4	18

The **graph** of $y = x^2 + 2$ moved up 2 units in comparison with $x = x^2$
Vertex $(0,2)$ **axis** of **symmetry** $x = 0$
Max $\{y = \infty\}$; **Min** $\{y = 2\}$

- The **graph** of a **relation** can be analyzed to determine if the **relation** is a **function**. Using a **vertical line** will determine if there are any corresponding values on the same **axis** ergo determining whether the **relation** is a **function** or not. If the **vertical line** cuts the **graph** more than once, it is not a **function**
- The standard **equation** of a **quadratic function**

Formula: $y = ax^2$
$a =$ **vertical** stretch or shrink

Formula: $y = x^2 + k$
$k =$ **vertical translation**

Formula: $y = ax^2 + k$
$a =$ **vertical** stretch or shrink
$k =$ **vertical translation**

- The larger a is, the narrower the **parabola** will be
- Identify all aspects of the following

Example: $y = 4x^2$
Vertex: $(0,0)$
Axis of **symmetry**: $x = 0(y - \text{axis})$
Max: $y = \infty$
Min: 0
Domain: $x \in \mathbb{R}$
Range: $y \geq 0, y \in \mathbb{R}$

- Expanded **quadratic function**

Formula: $y = a(x-h)^2 + k$

$a =$ **Vertical** stretch/shrink; if $a < 0$, opens downward on a **reflection** in x-**axis**

$(x-h) =$ **Horizontal translation**

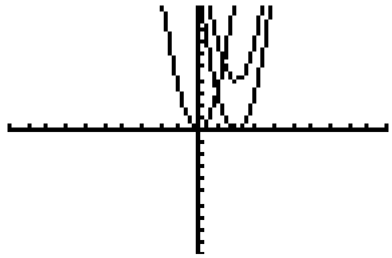
$k =$ **Vertical translation**

- To convert $y = x^2 \rightarrow y = a(x-h)^2 + k$, use the following table

Operation	Resulting Equation	Transforming
Multiply by $\boldsymbol{a}$	$y = ax^2$	Reflects in the x-axis, if $a < 0$ Stretches vertically (narrows), if $a > 1$ or a < -1 Shrinks vertically (widens), if $-1 < a < 1$
Replace $\boldsymbol{x}$ by $(\boldsymbol{x} - \boldsymbol{h})$	$y = a(x-h)^2$	Shifts h units to the right, if $h > 0$ Shifts h units to the left, if $k > 0$
Add $\boldsymbol{k}$	$y = a(x-h)^2 + k$	Shifts k units upward, if $k > 0$ Shifts k units downward, if $k < 0$

Property	Sign of a positive	Sign of a negative
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$x = h$
Direction of Opening	Up	Down
Comparison with $\boldsymbol{y = ax^2}$	Congruent	Congruent

Examples: Compare $y = 3x^2, y = 3(x-2)^2, y = 3(x-2)^2 + 4$

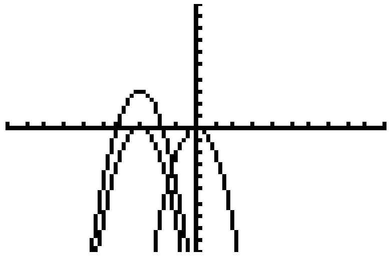


$y = 3x^2$
V: $(0,0)$
D: $x \in \mathbb{R}$
R: $y \geq 0$
Max: ∞
Min: 0
Up

$y = 3(x-2)^2$
V: $(2,0)$
D: $x \in \mathbb{R}$
R: $y \geq 0$
Max: ∞
Min: 0
Up

$y = 3(x-2)^2 + 4$
V: $(2,4)$
D: $x \in \mathbb{R}$
R: $y \geq 4$
Max: ∞
Min: 4
Up

Examples: Compare $y = -2x^2, y = -2(x+3)^2, y = -2(x+3)^2 + 3$



$y = -2x^2$
V: $(0,0)$
D: $x \in \mathbb{R}$
R: $y \leq 0$
Max: 0
Min: $-\infty$
Down

$y = -2(x+3)^2$
V: $(-3,0)$
D: $x \in \mathbb{R}$
R: $y \leq 0$
Max: 0
Min: $-\infty$
Down

$y = -2(x+3)^2 + 3$
V: $(-3,0)$
D: $x \in \mathbb{R}$
R: $y \leq 3$
Max: 0
Min: $-\infty$
Down

- **Graphing** $y = ax^2 + bx + c$ by completing the (**perfect**) **square**

Formula: $y = ax^2 + bx + c$

- To solve, get the value of half of b
- **Square** the value
- Make 2 instances of the value, one negative and one positive and place them within the formula following b
- **Factor equation**

Example:
$$y = x^2 + 8x + 7$$
$$y = x^2 + 8x + 4^2 - 4^2 + 7$$
$$y = x^2 + 8x + 16 - 16 + 7$$
$$y = (x + 4)^2 - 9$$

- The end result should be $y = a(x - h)^2 + k$
- If $k > 0, k$ is the **maximum** of the **function**
- If $k < 0, k$ is the **minimum** of the **function**
- **Axis** of **symmetry** is $-h$
- h, k is the **vertex point**

- Be aware that any **equation** can be **factored** by going into **fractions**

Example:
$$y = x^2 + 9x + 2$$
$$y = x^2 + 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 2$$
$$y = x^2 + 9x + \frac{81}{4} - \frac{81}{4} + 2$$
$$y = \left(x + \frac{9}{2}\right)^2 - \frac{81}{4} + 2$$
$$y = \left(x + \frac{9}{2}\right)^2 - \frac{81}{4} + \frac{8}{4}$$
$$y = \left(x + \frac{9}{2}\right)^2 - \frac{73}{4}$$

Example:
$$y = -\frac{1}{2}x^2 + \frac{3}{5}x + 9$$
$$y = -\frac{1}{2}\left(x^2 - \frac{6}{5}x\right) + 9$$
$$y = -\frac{1}{2}\left(x^2 - \frac{6}{5}x + \left(\frac{6}{10}\right)^2 - \left(\frac{6}{10}\right)^2\right) + 9$$
$$y = -\frac{1}{2}\left(x^2 - \frac{6}{5}x + \frac{36}{100} - \frac{36}{100}\right) + 9$$
$$y = -\frac{1}{2}\left(x - \frac{6}{10}\right)^2 + \frac{36}{200} + 9$$
$$y = -\frac{1}{2}\left(x - \frac{3}{5}\right)^2 + \frac{9}{50} + 9$$
$$y = -\frac{1}{2}\left(x - \frac{3}{5}\right)^2 + \frac{359}{50}$$
$$\mathrm{V} = \left(\frac{3}{5}, \frac{459}{50}\right)$$

- **Common factors** first

Example:
$$y = 2x^2 + 4x + 3$$
$$y = 2(x^2 + 2x) + 3$$
$$y = 2(x^2 + 2x + 1^2 - 1^2)3$$
$$y = 2((x + 1)^2 - 1) + 3$$
$$y = 2(x + 1)^2 - 2 + 3$$
$$y = 2(x + 1)^2 + 1$$

- Rearrange to solve

Example:
$$y = 10x - 5x^2$$
$$y = -5x^2 + 10x$$
$$y = -5(x^2 - 2x)$$
$$y = -5(x^2 - 2x + 1^2 - 1^2)$$

$y = -5(x-1)^2 + 5$

- Problem solving

Example: You have 600m of fence. You enclose a rectangular area. What dimensions yield the max area? What is the max area

$A = lw$
$600 = 2x + 2y$
$600 - 2x = 2y$
$\frac{600 - 2x}{2} = y$
$y = 300 - x$
$\therefore A = lw$
$A = x(300 - x)$
$A = 300x - x^2$
$A = -x^2 + 300x$
$A = -(x^2 - 300x)$
$A = -(x^2 - 300x + 150^2 - 150^2)$
$A = -(x - 150)^2 + 150^2$
$A = -(x - 150)^2 + 22500$
$\mathrm{V} = (150{,}22500)$
$\therefore x = 150$
$y = 300 - 150$
$y = 150$
150×150
$A = 22500\mathrm{m}^2$

Solving Quadratic Equations

- Solving an **equation** means finding values(s) for x
- Solve the **bracketed terms** to find both **intercepts points** of the **parabola**

Example:

$$y = 2x^2 - x - 3; \text{Adds: } -1, \text{Multiplies: } -6$$
$$y = 2x^2 + 2x - 3x - 3$$
$$y = 2x(x+1) - 3(x+1)$$
$$y = (2x-3)(x+1)$$
$$2x - 3 = 0$$
$$2x = 3$$
$$x = \frac{3}{2}$$
$$x + 1 = 0$$
$$x = -1$$

- Get the whole **equation** on one side to create a **quadratic function** and solve for x
- Solve for the **square** of the **function** to find it's **vertex**
- Find it's y-int by **substituting** 0 for x

Example:

$$x^2 - 3x = -2$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x - 1 = 0$$
$$x = 1$$
$$x - 2 = 0$$
$$x = 2$$
$$y = x^2 - 3x + 2$$
$$y = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2$$
$$y = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{8}{4}$$
$$y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$
$$\text{V} = \left(\frac{3}{2}, -\frac{1}{4}\right)$$
$$y\text{-int} = 0^2 - 3(0) + 2$$
$$y\text{-int} = 2 = (0,2)$$

- **Common factors** first

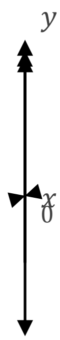
Example:

$$4x^2 \ + 3x = 0$$
$$x(4x+3) = 0$$
$$x = 0$$
$$4x + 3 = 0$$
$$x = -\frac{3}{4}$$
$$4x^2 + 3x = 0$$
$$4\left(x^2 + \frac{3}{4}x\right) = 0$$
$$4\left(x^2 + \frac{3}{4}x + \frac{9}{64} - \frac{9}{64}\right) = 0$$
$$4\left(x + \frac{3}{8}\right)^2 - \frac{36}{64} = 0$$
$$\text{V} = \left(-\frac{3}{8}, -\frac{36}{64}\right) = \left(-\frac{3}{8}, -\frac{9}{16}\right)$$
$$y\text{-int} = 0$$

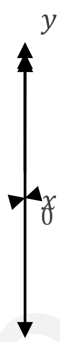
Example:

$$\frac{x^2}{9} - \frac{x}{3} = 2$$
$$9\left(\frac{x^2}{9} - \frac{x}{3}\right) = 9(2)$$
$$x^2 - 3x = 18$$
$$x^2 - 3x - 18 = 0$$
$$(x-6)(x+3)$$
$$x - 6 = 0$$
$$x = 6$$
$$x + 3 = 0$$
$$x = -3$$
$$x^2 - 3x - 18 = 0$$
$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 18 = 0$$
$$\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{72}{4} = 0$$
$$\left(x - \frac{3}{2}\right)^2 - \frac{81}{4} = 0$$
$$\text{V} = \left(\frac{3}{2}, -\frac{81}{4}\right)$$
$$y\text{-int} = \frac{0}{9}2; (0, -2)$$

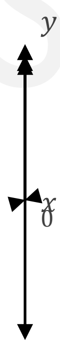
- There are only 3 possible outcomes



Example: 2 distinct **real roots**



Example: No **real roots**



Example: 2 equal **real roots**

- When given the **vertex** of a **parabola**, you can find the y-int by **substituting** the **vertex** into the **quadratic function** in place of x

Example:

$$\text{Vertex} = 1$$
$$y = 1^2 - 2(1) - 8$$
$$y = 1 - 2 - 8$$
$$y = -9$$

Quadratic Formula

- The formula is used to determine the x **variable** for certain conventional methods will not render the correct value

Formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- When given the standard **quadratic function**; to proove the formula, you must follow a series of steps

Example: Given: $ax^2 + bx + c = 0$

- Complete the **square**

Example:

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c = 0a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2}\right) + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

- Isolate x

Example:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- Solving x

Example: $8x^2 + 6x - 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-6 \pm \sqrt{6^2 - 4(8)(-9)}}{2(8)}$$
$$x = \frac{-6 \pm \sqrt{36^2 - 288}}{16}$$
$$x = \frac{-6 \pm \sqrt{324}}{16}$$
$$x = \frac{-6 \pm 18}{16}$$
$$x = \frac{-3 \pm 9}{8}$$
$$x = \frac{-3 + 9}{8}$$
$$x = \frac{6}{8}$$
$$x = \frac{3}{4}$$
$$x = \frac{-3 - 9}{8}$$
$$x = -\frac{12}{8}$$
$$x = -\frac{3}{2}$$

These are **rational roots**

Example: $x^2 - 3x - 1 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{3 \pm \sqrt{13}}{2}$$
$$x = \frac{3 + \sqrt{13}}{2}$$
$$x = \frac{3 - \sqrt{13}}{2}$$

These are **irrational roots**

Example: $x^2 - 2x + 3 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$

$$x = \frac{2 \pm \sqrt{-8}}{2}$$

There are no **real** solutions

- There are several possible outcomes

Example: If $b^2 - 4ac < 0$ (Negative): No **real** solutions

If $b^2 - 4ac = 0$: 2 **real equal roots** (**double root**)

If $b^2 - 4ac = \text{perfect square}$: 2 **real** distinct **roots** (**rational**)

If $b^2 - 4ac > 0$ (Not a perfect square): 2 **real** distinct **roots** (**irrational**)

- When there is 2 **variables**, remove the x **term** and leave any other

Example: $x^2 + 2xy - y^2 = 0$

$$x = \frac{-2y \pm \sqrt{(2y)^2 - 4(1)(-y^2)}}{2(1)}$$

$$x = \frac{-2y \pm \sqrt{4y^2 + 4y^2}}{2}$$

$$x = \frac{-2y \pm \sqrt{8y^2}}{2}$$

$$x = \frac{-2y \pm \sqrt{4}\sqrt{2}\sqrt{y^2}}{2}$$

$$x = \frac{-2y \pm 2\sqrt{2}\sqrt{y^2}}{2}$$

$$x = \frac{2(-y \pm \sqrt{2}\,y)}{2}$$

$$x = -y \pm \sqrt{2}y$$

- Determining the **minimum** and **maximum** can be done by **completing the square** of **quadratic functions**
- Where there is no **coefficient**, **add** and **subtract** the **square** of half the **coefficient** of x
- Group the **perfect square trinomial**
- Simplify the **trinomial** as a **square binomial**

Example:

$$y = x^2 + 12x - 7$$
$$y = x^2 + 12x + 36 - 36 - 7$$
$$y = (x^2 + 12x + 36) - 36 - 7$$
$$y = (x + 6)^2 - 43$$
$$\therefore \text{Min} = -43, x = -6$$

- When given a **coefficient** next to x, group the containing **terms** of x
- Factor first to **terms** only containing x
- Simplify the **trinomial** as a **square binomial**

Example:

$$y = 4x^2 - 24x + 31$$
$$y = (4x^2 - 24x) + 31$$
$$y = 4(x^2 - 6x + 9 - 9) + 31$$
$$y = 4[(x - 3)^2 - 9] + 31$$
$$y = 4(x - 3)^2 - 36 + 31$$
$$y = 4(x - 3)^2 - 5$$
$$\therefore \text{Min} = -5, x = 3$$

- Not all **functions** have **perfect square integers**, therefore it may involve **fractions**

Example:

$$y = 5x - 3x^2$$
$$y = -3x^2 + 5x$$
$$y = -3\left(x^2 - \frac{5}{3}x\right)$$
$$y = -3\left(x^2 - \frac{5}{3}x - \frac{25}{36} + \frac{25}{36}\right)$$
$$y = -3\left[\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right]$$
$$y = -3\left(x - \frac{5}{6}\right)^2 + \frac{25}{36}$$
$$\therefore \text{Max} = \frac{25}{12}, x = \frac{5}{6}$$

- Solving **quadratic equations** through **factoring**

Example:
$$x^2 - 6x - 27 = 0$$
$$(x-9)(x+3) = 0$$
$$x = 9, -3$$
$$\text{Midpoint} = \frac{9 + (-3)}{2} = 3$$
$$V = 3^2 - 6(3) - 27$$
$$V = (3, -36)$$

- Solve by completing the **square**

Example:
$$2x^2 - 5x - 1 = 0$$
$$2\left(x^2 - \frac{5}{2}x\right) - 1 = 0$$
$$2\left(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}\right) - 1 = 0$$
$$2\left(\left[x - \frac{5}{4}\right]^2 - \frac{25}{16}\right) - 1 = 0$$
$$2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} - \frac{8}{8} = 0$$
$$2\left(x - \frac{5}{4}\right)^2 - \frac{33}{8} = 0$$
$$2\left(x - \frac{5}{4}\right)^2 = \frac{33}{8}$$
$$\left(x - \frac{5}{4}\right)^2 = \frac{33}{16}$$
$$x - \frac{5}{3} = \pm\sqrt{\frac{33}{16}}$$
$$x - \frac{5}{4} = \frac{\pm\sqrt{33}}{4}$$
$$x = \frac{\pm\sqrt{33}}{4} + \frac{5}{4}$$
$$x = \frac{\pm\sqrt{33} + 5}{4}$$

- Solve by **quadratic formula**

Example: $x^2 + 5x + 3 = 0$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{-5 \pm \sqrt{25 - 12}}{2}$$
$$x = \frac{-5 \pm \sqrt{13}}{2}$$

- Applications to **quadratic functions**

Example: A football is punt into the air. Its height, h, in meters, after t seconds y:

$h = -5t^2 + 30t$
$h = -5(t^2 - 6t)$
$h = (t^2 + 6t + 9 - 9)$
$h = -5(t - 3)^2 + 45$
$\therefore \text{Max} = 45\text{m}$
$0 = -5t^2 - 30t$
$0 = -5t(t - 6)$
$\therefore t = 0, t = 6$

Example: A CD player sells for \$6000 Sales average 80 per month. Every \$100 increase there will be 1 less CD player sold.

Let $x =$ every \$100 increase
Let $r =$ Revenue
$r = (6000 + 100x)(80 - x)$
$r = 480000 + 200x - 100x^2$
$r = -100x^2 + 2000x + 480000$
$r = -100(x^2 + 20x) + 480000$
$r = -100(x^2 + 20x + 100 - 100) + 480000$
$r = -100[(x - 10)^2) - 100] + 480000$
$r = -100(x - 10)^2 + 490000$
$\therefore$ Max revenue $= \$490000$

Example: A rectangle lawn, $7\text{m} \times 5\text{m}$. Uniform boarded of flowers is planted along 2 adjacent sides. If flowers cover 6.25m^2, how wide is boarder.

Let $x =$ width of boarder

$A = 41.25\text{m}^2$

$(5 + x)(7 + x) = 41.25$

$35 + 12x + x^2 = 41.25$

$x^2 + 12x - 6.25 = 0$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(-6.25)}}{2}$$

$$x = \frac{-12 \pm \sqrt{169}}{2}$$

$$x = \frac{-12 \pm 13}{2}$$

$$x = -\frac{25}{2}, \frac{1}{2}$$

$\therefore$ Border $= 0.5\text{m}$ wide

- Evaluating for **function** notation
- $f(x)$ means **Function** of x
- Used to fine when x equals an **integer**

Example: $y = 2x^2 + 3x - 4$

$f(x) = 2x^2 + 3x - 4$

Example: $f(x) = 5x - 2$

$f(3) = 5(3) = 2$

$f(3) = 13$

Example: $f(0) = -2$

Example: Find x if $f(x) = 4x + 3$

$f(x) = 8$

$8 = 4x + 3$

$$x = \frac{5}{4}$$

Example: Find x if $f(x) = 4x + 3$

$f(x) = 0$

$0 = 4x + 3$

$$x = -\frac{3}{4}$$

Exponential Functions

- **Functions** that either have an **exponential** growth $(a > 1)$ or **exponential** decay $(0 < a < 1)$ where c is the initial value, a is the growth or decay **factor**, and x is the measure of time

  Formula: $f(x) = c(a)^x$

- Use a **table** of values to express and **graph** the formula

  Example: Bactereia doubles each hour and you start with 35 cells, how many in 3 hours?
  $\therefore f(x) = 35(2)^x$
  $f(3) = 35(2)^3$
  $f(3) = 280$

  Example: Deer population is 80% of what it was each year and you start with 15000, how many remin in 12 years?
  $\therefore f(x) = 15000(0.8)^x$
  $f(12) = 15000(0.8)^{12}$
  $f(12) = 1030$

  Example: $5200 doubles every 6 years. Find the growth after 15 years
  $\therefore f(x) = 5200(2)^x$
  $x = \frac{15}{6} = 2.5$
  $f(15) = 5200(2)^{2.5}$
  $f(15) = 29415.64$

- The **domain** value (x) will be greater than 0
- The **range** value (y) will be greater than 0 and in certain cases may have an **asympototes** of 0

  Example: $y = 2500(0.25)^x$
  $D: \{x \geq 0\}$
  $R: \{y \geq 0 \mid y \neq 0\}$

Transformations

Horizontal and vertical translations of functions

- A **Translation** is when something is shifted or moved
- Can be calculated through 2 additional **variables**, p and q
- Horizontal **translations** are determined by q. It is located outside the **function** expression
- A vertical **translation** is determined by p. It is located inside the **function expression**. Also, the value is always opposite its display value
- If either q and p are not present, then it is only a **translation** on one **axis**

Formula: $y = f(x \pm p) \pm q$

Example: $y = \sqrt{x-1} + 1$
Move right 1, up 1

Example: $y = \frac{1}{x+4} - 2$
Move left 4, down 2

- Asymptotes are dotted lines that appear on a graph in a function such as $y = f\left(\frac{1}{x}\right)$. These lines indicate that the function never reaches the asymptotes, in this case, 0

Reflections of functions

- Whichever axis is being **reflected**, the formula must apply
- For a **reflection** on the x axis

Formula: $-f(x)$

Example:
$f(x) = 6x - 1$
$-f(x) = -(6x - 1)$
$-f(x) = -6x + 1$

Example:
$f(x) = -5x^2 + 3$
$-f(x) = -(-5x^2 + 3)$
$-f(x) = 5x^2 - 3$

- For a **reflection** on the y axis

Formula: $f(-x)$

Example:
$f(x) = \sqrt{-x + 3}$
$f(-x) = \sqrt{x(-x) + 3}$
$f(-x) = \sqrt{x + 3}$

Example:
$f(x) = (x - 4)^2$
$f(x) = x^2 - 8x + 16$
$f(-x) = x^2 + 8x + 16$
$f(-x) = (x + 4)^2$

- An **invariant** point Is a point that lies on the axis line and does not shift after a **reflection**

The Inverse of a function

- x and y values are swapped
- When both the original and **inverse function** are **graphed**, the hypothetical **reflection line** is $y = x$
- To find, swap the **variables** and then solve for y

Formula: $f^{-1}(x)$

Example:
$f(x) = 3x + 6$
$y = 3x + 6$
$x = 3y + 6$
$\frac{x-6}{3} = \frac{3y}{3}$
$y = \frac{x-6}{3}$
$f^{-1}(x) = \frac{x-6}{3}$
The **invariant point** is $(-3, -3)$

Example:
$f(x) = x^2 - 9$
$y = x^2 - 9$
$x = y^2 - 9$
$x + 9 = y^2$
$y = \pm\sqrt{x+9}$
$D_f = \{x \epsilon \mathbb{R}\}$
$R_f = \{y \epsilon \mathbb{R} | y \geq -9\}$
$D_{f^{-1}} = \{x \epsilon \mathbb{R} | x \geq -9\}$
$R_{f^{-1}} = \{y \epsilon \mathbb{R}\}$

- Not all **inverse functions** will prove to be a real **function**, therefore, take the positive end of the **function** by **restricting** the **domain**

Example:

$f(x) = (x+5)^2$
$y = (x+5)^2$
$x = (y+5)^2$
$\pm\sqrt{x} = y + 5$
$y = \pm\sqrt{x} - 5$
$D_f = \{x \in \mathbb{R} | x \geq -5\}$
$R_f = \{y \in \mathbb{R} | y \geq 0\}$
$D_{f^{-1}} = \{x \in \mathbb{R} | x \geq 0\}$
$R_{f^{-1}} = \{y \in \mathbb{R} | y \geq -5\}$

Applications for inverse functions

- When working with alternative **variables**, there is no need to swap variables, just solve for the **isolated term**

Example: The cost of renting a car for a day is a flat rate of $60 and $\frac{\$0.35}{\text{km}}$

Let $d = \#\text{ of km}$

Let $c = \text{cost}$

$y = 0.35x + 60$

$f(x) = 0.35x + 60$

$c(d) = 0.35d + 60$

$d = \dfrac{c - 60}{0.35}$

$D_f = \{d\, \epsilon \mathbb{R} | x \geq 0\}$

$R_f = \{c \epsilon \mathbb{R} | y \geq 60\}$

$D_{f^{-1}} = \{c \epsilon \mathbb{R} | x \geq 60\}$

$R_{f^{-1}} = \{d \epsilon \mathbb{R} | y \geq 0\}$

Vertical and horizontal stretches of functions

- Recall the effect a in a **parabola** $y = ax^2$
- **Vertical expansion**: If $a > 1$
- **Vertical compression**: if $0 < a < 1$
- With **vertical stretches**, points on the x axis are **invariant**
- In **point** (x, y) on $y = f(x)$ becomes (x, ay) on $y = af(x)$
- Therefore, the **domain** will remain the same while the **range** changes based on the **multiple**

Example: (Each case involves a set of **ordered pairs**, watch **domain**)

Cases: $y = 2f(x), y = f(x), y = \frac{1}{2}f(x)$

$D_{f(x)} = \{x\epsilon\mathbb{R}| - 3 \leq x \leq 3\}$

$R_{f(x)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 4\}$

$D_{2f(x)} = \{x\epsilon\mathbb{R}|-3 \leq x \leq 3\}$

$R_{2f(x)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 8\}$

$D_{\frac{1}{2}f(x)} = \{x\epsilon\mathbb{R}|-3 \leq x \leq 3\}$

$R_{\frac{1}{2}f(x)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 2\}$

- Recall the effect of k in a **parabola** $y = f(kx)$
- **Horizontal expansion**: If $0 < k < 1$
- **Horizontal compression**: if $k > 1$
- With **horizontal stretches**, points on the y axis are **invariant**
- In **point** (x, y) on $y = f(x)$ becomes $(\frac{x}{k}, y)$ on $y = f(kx)$
- Therefore, the **range** will remain the same while the **domain** changes based on the **multiple**

Example: (Each case involves a set of **ordered pairs**, watch **domain**)

Cases: $y = f(2x), y = f(x), y = f\left(\frac{1}{2}x\right)$

$D_{f(x)} = \{x\epsilon\mathbb{R}| - 2 \leq x \leq 4\}$

$R_{f(x)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 4\}$

$D_{f(2x)} = \{x\epsilon\mathbb{R}|-1 \leq x \leq 2\}$

$R_{f(2x)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 4\}$

$D_{f\left(\frac{1}{2}x\right)} = \{x\epsilon\mathbb{R}|-4 \leq x \leq 8\}$

$R_{f\left(\frac{1}{2}x\right)} = \{y\epsilon\mathbb{R}|0 \leq x \leq 4\}$

- When working with a **radical function**, be aware of the way it is graphed

Example: $y = \sqrt{2x}$ is the **graph** of $y = \sqrt{x}$ compressed **horizontally** by a **factor** of $\frac{1}{2}$

$y = \sqrt{\frac{1}{2}x}$ is the **graph** of $y = \sqrt{x}$ expanded **horizontally** by a **factor** of 2

- When a **function stretches** both **horizontally** and **vertically**, the **stretches** can be performed in either order to get the same result

Example: Given$y = f(x)$, **graph** $y = 3f\left(\frac{1}{2}\,x\right)$

The **point** (x, y) on $y = f(x)$ becomes $(\frac{x}{k}, ay)$ on $y = af(kx)$

Combinations of transformations

- When performing combinations of **transformations**, work in this recommended order: **Expansions** and **compressions**, **reflections**, and **translations**
- Describe the **transformations** of the following **functions**

Formula: $y = af[(k(x-p)] + q$

$a =$ **Amplitude**, **compression**, **vertical stretch factor/size**, negative **reflects** on $x, (a < 0)$

$k =$ **Reciprocal** of **horizontal stretch** factor/size, negative **reflects** on $y, (k < 0)$

Period from $\frac{360°}{k}$

$p =$ Negative of the **horizontal** shift (backwards rule)/location

$p > 0 = \text{Right } (x \rightarrow \ -\#°)$

$p < 0 = \text{Left } (x \rightarrow \ +\#°)$

$q =$ **Vertical** shift/location

$q > 0 = \text{Up}$

$q < 0 = \text{Down}$

Example: Given $f(x) = -4(x-2)^2 + 3$

Reflect on x axis, translate right 2 and up 3, expand vertically by a factor of 4

Example: Given $f(x) = -\left(\frac{2}{x-1}\right)$

Reflect on x axis, translate right 1, expand vertically by a factor of 2; asymptotes: $x = 1, y = 0$

Example: Given $g(x) = \sqrt{2x+8} \rightarrow \sqrt{2(x+4)}$

Translate left 4, expand vertically by a factor of 2

Example: Given $h(x) = \sqrt{\left(-\frac{1}{2}\right)x + 3} \rightarrow \sqrt{-\frac{1}{2}(x-6)}$

Reflect on y axis, translate right 6, compress vertically by a factor of $\frac{1}{2}$

Example: Given $y = f(3x)$

Horizontal compression by a factor of $\frac{1}{3}$

Example: Given $y = 3f\left(\frac{1}{2}x\right)$

Vertical expansion by a factor of 3, horizontal expansion by a factor of 2

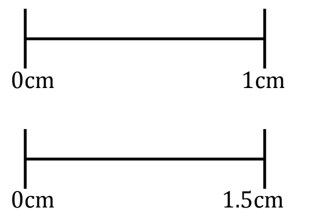
Geometry

Linear Measurements

Certain plus 1

- Measure all digits plus one estimated value
- If the measurement is right on the line the last digit is a zero (0)

Example:



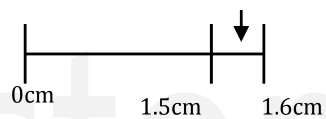
1.0 read from ruler

$1.0(0)$ *estimated digit*

$1.5(0)$ *estimated digit*

- If the measurement ends between the readings estimate the last digit $(1-9)$

Example:



$1.5(5)$ *estimated digit*

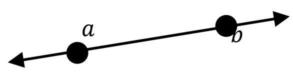
Lines

Notations

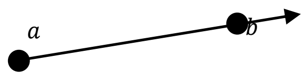
- **Anchor points** or **endpoints** appear as dots and arrows determine if the **line** extends in that direction forever

Types

- A **line** is a straight path of **points** that extends forever (∞) in both directions

Example: 

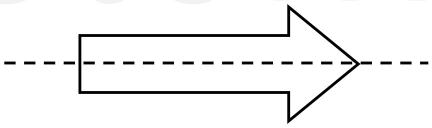
- A **ray** is a part of a line that begins at 1 **endpoint** and extends forever in one direction

Example: 

- A **line segment** is a part of a **line** and has 2 **endpoints**

Example: 

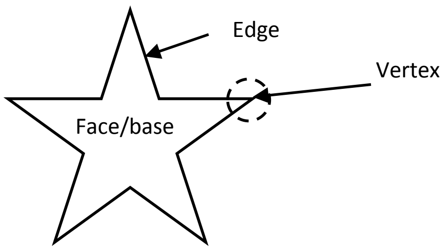
- The **line** of symmetry is when a shape can be split in half and share identical/mirroring properties

Example: 

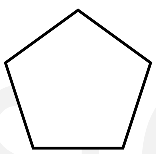
Polygons

A **polygon** or shape 2 dimensional

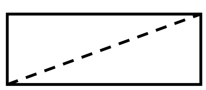
- A **polygon** is a 2-dimensional closed figure made up of **line segments**
- A **vertex** is where 2 or more **edges/endpoints** meet. Encloses **area** of **point** and creates an **angle**
- An **edge** is more like a **line segment** but the difference is that for a **line segment** to become an **edge**, the shape has to be enclosed
- A **face** can only be formed when a shape enclosed by **edges**. The insides of the **edges** is the **face**

Example: 

- **Regular polygons** are **polygons** that have all equal **sides** and **angles**

Example: 

- **Regular polygons** are given names for the number of **sides** they have (refer to chart at the end of this section)
- A **convex polygon** is a **polygon** that has **line segments** within the polygon

Example: 

- A **concave polygon** is a **polygon** that has **line segments** outside the **polygon**

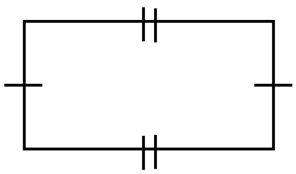
Example:

Types of polygons

- **Polygons** are classified and named by the number of **sides** it has
- **Polygons** with 3 **sides** are classified as **triangles**
- **Polygons** with 4 **sides** are classified as **quadrilaterals**
- **Polygons** with more than 4 **sides** are classified as **polygons**

Notation

- **Sides** that share equal properties will sometimes be labelled with a accents (**lines**, arrows, etc.) sets of these accents are paired with no less than 2

Example: 

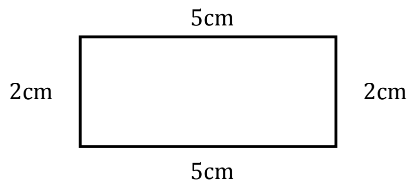
Regular polygon chart

Name	Sides	Angle (Total) $180(n-2)$	Angle(Individual) $\frac{180(n-2)}{n}$
Henagon	1	$Undefined$	$Undefined$
Digon	2	$0°$	$0°$
Triangle	3	$180°$	$60°$
Square	4	$360°$	$90°$
Pentagon	5	$540°$	$108°$
Hexagon	6	$720°$	$120°$
Heptagon	7	$900°$	$128.5°$
Octagon	8	$1080°$	$135°$
Nonagon	9	$1260°$	$140°$
Decagon	10	$1440°$	$144°$
Hendecagon	11	$1620°$	$147.27°$
Dodecagon	12	$1800°$	$150°$
Icosagon	20	$3240°$	$162°$
Chiliagon	1000	$179640°$	$179.64°$
Myriagon	10000	$1799640°$	$179.964°$
Googolgon	10^{100}	$1.8 \times 10^{102°}$	$180°$

Perimeter

The **perimeter** of a figure is the distance around it

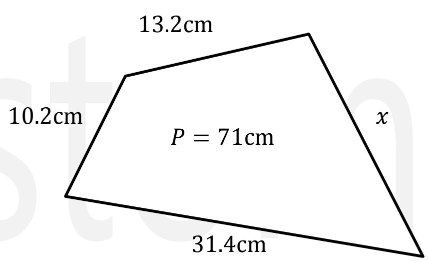
- The **sum** of the length of **sides** on an object produces its **perimeter**



Example:

$$P = l + w + l + w$$
$$P = 2l + 2w$$
$$P = 5 + 2 + 5 + 2$$
$$P = 14\text{cm}$$

- When given the total **perimeter** and there is only 1 missing **side**, **sum** up the **sides** given and **subtract** it against the total **perimeter** to find the missing value

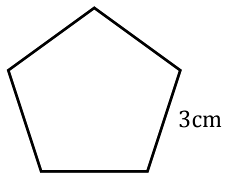


Example:

$$x = (13.2 + 10.2 + 31.4) - P$$
$$x = 54.8 - 71$$
$$x = 16.2\text{cm}$$

- If the figure is a **regular polygon**, than each side is equal therefore 1 measure is only required

Example:



$P = s \times 5$
$P = 3 \times 5$
$P = 15\text{cm}$

- When given an **regular polygon** and the **perimeter** and you are asked to find the length of 1 **side**; simply correspond the name of the polygon to a number and divide the **perimeter** with the **polygon**

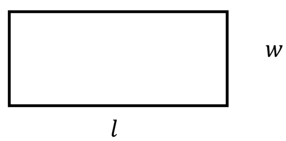
Example: Pentagon (5); $P = 45\text{cm}$
$s = 5 \div 45 = 9$

Area

The space inside an object

Rectangles

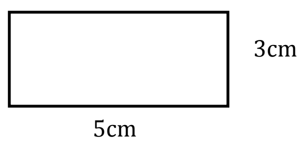
- To calculate the **area** of a **rectangle**, simply multiple the length and width



Formula:

$A = l \times w$ or lw

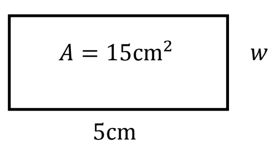
- When working with **area** and units be sure to square the end result by placing it as units squared (^{2}) since the formula works on a 2 dimensional basis



Example:

$A = lw$
$A = 5 \times 3$
$A = 15\text{cm}^2$

- When given the total **area** and one **side**; to solve for the missing value simply **divide** the **area** to the **side** given



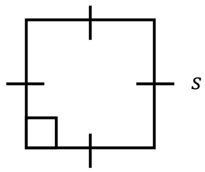
Example:

$w = A \div l$
$w = 15 \div 5$
$A = 3\text{cm}$

Squares

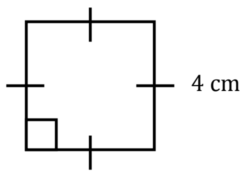
- To calculate the **area** of a square, simply multiple the **side** given

Formula:



$A = s^2$

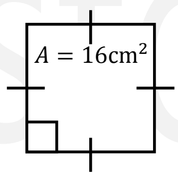
Example:



$A = s^2$
$A = 4^2$
$A = 16\text{cm}^2$

- When given the total **area** and you need to solve for the **side** value, simply **divide** the total **area** by 4

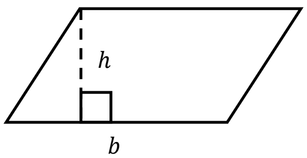
Example:



$s = A \div 4$
$s = 16 \div 4$
$s = 4\text{cm}$

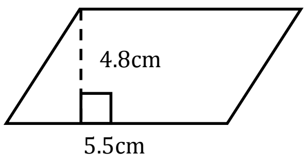
Parallelogram

- To calculate the **area** of a parallelogram, simply **multiply** its base and height



Formula:

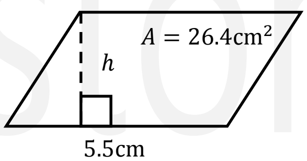
$A = b \times h$ or bh



Example:

$A = b \times h$
$A = 5.5 \times 4.8$
$A = 26.4\text{cm}^2$

- When given the total **area** and the height or base, to solve for the missing value simply **divide** the **area** by the measure given

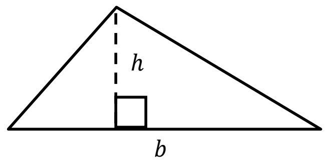


Example:

$h = A \div b$
$h = 26.4 \div 5.5$
$h = 4.8\text{cm}$

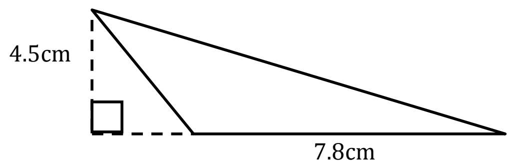
Triangle

- To calculate the **area** of a triangle, simply **multiply** its base and height and **divide** the **sum** by 2



Formula:

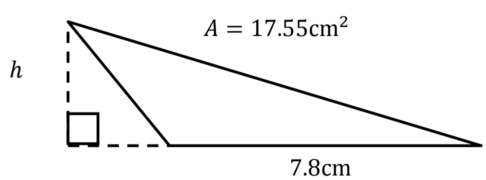
$$A = \frac{1}{2} b \times h \text{ or } \frac{bh}{2}$$



Example:

$$A = \frac{bh}{2}$$
$$A = \frac{7.8 \times 4.5}{2}$$
$$A = \frac{35.1}{2}$$
$$A = 17.55\text{cm}^2$$

- When given the total **area** and the height or base, to solve for the missing value simply **divide** the **area** by the measure given and double the **quotient**



Example:

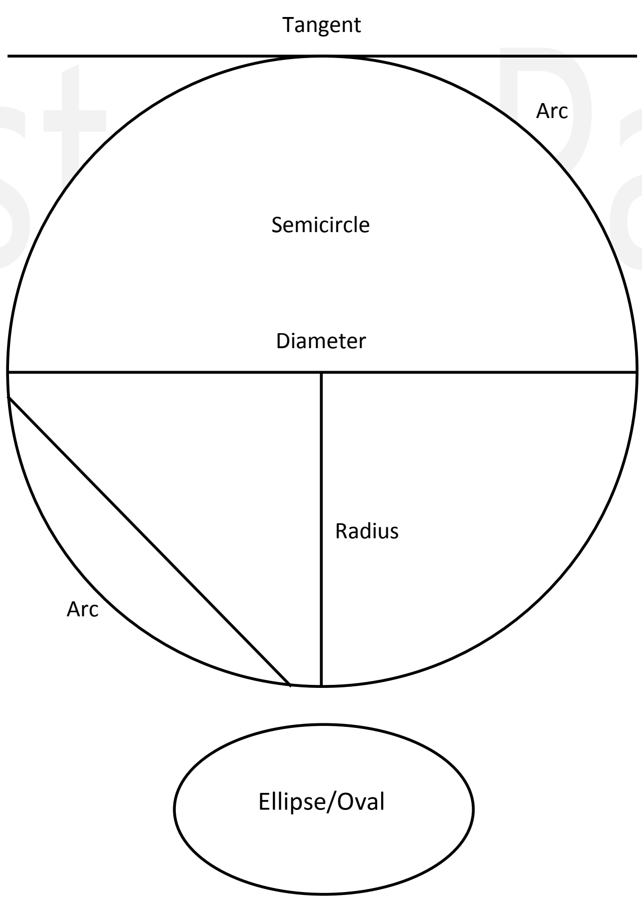
$$h = (A \div b) \times 2$$
$$h = (17.55 \div 7.8) \times 2$$
$$h = 2.25 \times 2$$
$$h = 4.5\text{cm}$$

Circle

The only **polygon** without a **vertex** or **vertices** and only 1 **edge**

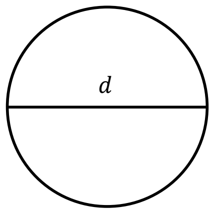
- **Circumference** is the distance around a circle
- **Diameter** is a **line segment** that joins 2 parts on a circle and passes through the center; double the **radius**
- **Radius** is the distance from the center of the circle to the **edge**; half of the **diameter**
- **Chord** is a **line segment** that joins any 2 **points** on the **circumference** of a circle
- **Arc** is a section of a **circumference** of a circle that lies between 2 ends of a **chord** therefore there are always 2 **arcs** on a circle on either **side** of the **chord**
- **Semicircle** is half of a whole circle
- **Tangent** is where the circle meets an **edge** and follows through **perpendicular**
- **Pi/π** is the amount of times the **diameter** can fit around the **circumference** of a circle. When the **diameter** is 1, π is 3.141592654 or 3.14

Diagram



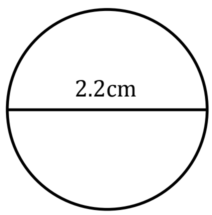
Perimeter of a circle

- The **perimeter** of a circle is also referred to as the **circumference**
- The distance across a circle through the center of a circle is called the **diameter**
- To calculate the **circumference** of a circle, simply **multiply** π and the **circumference**



Formula:

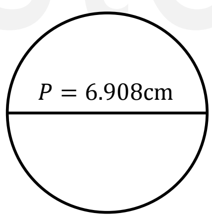
$P = \pi \times d$ or πd



Example:

$P = \pi d$
$P = 3.14 \times 2.2$
$P = 6.908\text{cm}$

- When given the total **circumference**; to solve the **diameter** simply **divide** the **perimeter** by π

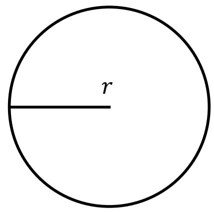


Example:

$d = P \div \pi$
$d = 6.908 \div 3.14$
$d = 2.2\text{cm}$

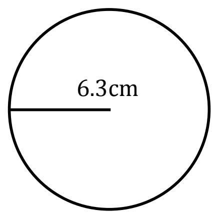
Area of a circle

- The distance from the center to the **circumference** is called the **radius**
- To calculate the **area** of a circle, simply **multiply** π and the **radius squared**



Formula:

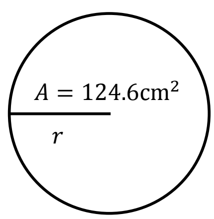
$$A = \pi r^2$$



Example:

$$A = \pi r^2$$
$$A = 3.14 \times 6.3^2$$
$$A = 3.14 \times 39.96$$
$$A = 124.6\text{cm}^2$$

- When given the total **area**; to solve the **radius** simply **divide** the **area** by π and find the **square root** of the **quotient**

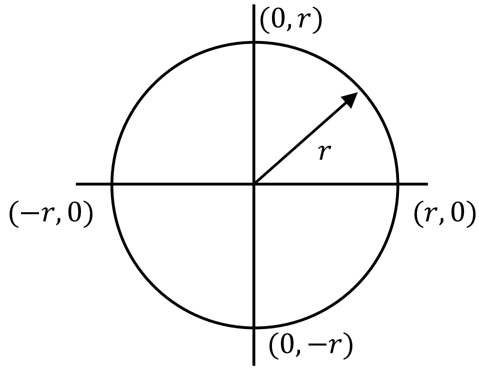


Example:

$$r = \sqrt{A \div \pi}$$
$$r = \sqrt{124.6 \div 3.14}$$
$$r = \sqrt{39.96}$$
$$r = 6.3\text{cm}$$

Graphing a circle

- The center of the **circle** can be represented by a pair of **coordinates**
- A **circle** can be represented by this formula



Formula: $x^2 + y^2 = r^2$

Example: $x^2 + y^2 = 25$
$r^2 = 25$
$r = 5$

- Equation of a circle with center (p, q)

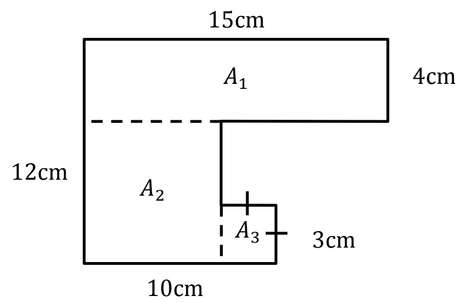
Formula: $(x - p)^2 + (y - q)^2 = r^2$

Example: $(x - 5)^2 + (y - 2)^2 = 25$
Center is $(5, -2)$

Area of Composite Figures

A **composite figure** is when multiple **polygons** are meshed together to create a new shape

- To find the **area** of a composite figure you must first **divide** the figure into regular identifiable shapes and then apply the formula to solve
- Be sure to correspond to your division of **area**



Example:

$A_1 = lw$
$A_1 = 15 \times 4$
$A_1 = 60\text{cm}^2$

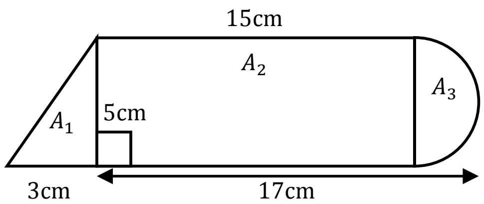
$A_2 = lw$
$A_2 = (12 - 4) \times (10 - 3)$
$A_2 = 8 \times 7$
$A_2 = 56\text{cm}^2$

$A_3 = s^2$
$A_3 = 3^2$
$A_3 = 9\text{cm}^2$

$A = A_1 + A_2 + A_3$
$A = 60 + 56 + 9$
$A = 125\text{cm}^2$

- Even when working with different **polygons**, just correspond to the sectors you make

Example:



$$A_1 = \frac{bh}{2}$$
$$A_1 = \frac{3 \times 5}{2}$$
$$A_1 = \frac{15}{2}$$
$$A_1 = 7.5\text{cm}^2$$

$$A_2 = lw$$
$$A_2 = 15 \times 5$$
$$A_2 = 75\text{cm}^2$$

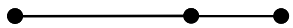
$$A_3 = \pi r^2$$
$$A_3 = 3.14 \times (17 - 15)^2$$
$$A_3 = 3.14 \times 2^2$$
$$A_3 = 3.14 \times 4$$
$$A_3 = 12.56\text{cm}^2$$

$$A = A_1 + A_2 + A_3$$
$$A = 7.5 + 75 + 12.56$$
$$A = 95.06\text{cm}^2$$

Angles

Formation

- An **angle** is formed by either 2 **rays** of **line segments** with a common **endpoint** called a **vertex**



Example:

Types

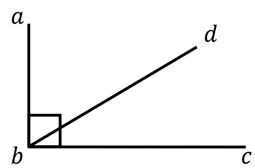
- **Acute angle**: $< 90°$
- **Obtuse angle**: $> 90°$
- **Right angle**: $= 90°$
- **Straight angle**: $= 180°$
- **Reflex angle**: $> 180°$ and $< 360°$

Certain angles have relationships

- Several types of **angle** relationships
- A **complementary angle** is when **angles add** up to 90°

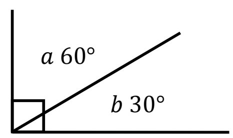
Complementary

Formula:



$\angle abd + \angle dbc = 90°$

Example:

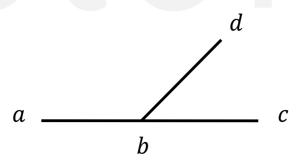


$a + b = 90°$
$60° + 30° = 90°$
$a = 90° - 60° = 30° = b$
$b = 90° - 30° = 60° = a$

- A **supplementary angle** is when **angles add** up to 180°

Supplementary

Formula:



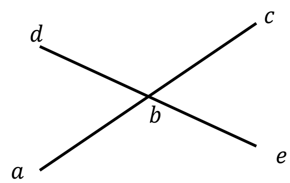
$\angle abd + \angle dbc = 180°$

Example:

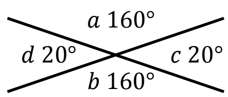
$a + b = 180°$
$120° + 60° = 180°$
$a = 180° - 120° = 60° = b$
$b = 180° - 60° = 120° = a$

- An **opposite angle** is when **angles** opposite of each other are equal and all **add** up to 360°

Opposite



Formula: $\angle abd = \angle cbe$



Example:

$a + b + c + d = 360°$
$160° + 160° + 20° + 20° = 360°$
$a = b; a + b = 320°; 360° - 320° = 40° = c, d$
$c = d; c + d = 40°; 40° \div 2 = 20° = c, d$

Transversal means when a **line** or **line segment** is crossing 2 or more **lines**

- When a **transversal** crosses 2 **parallel lines**, the **alternate angles** are equal, the **corresponding angles** are equal, and the **co-interior angles add** to 180°

Formula: t = **transversal** line

$\angle a = \angle b$
$\angle b = \angle d$
$\angle b + \angle c = 180°$

- **Adjacent** means adjoining or next to

Example: a and b are **adjacent** sides

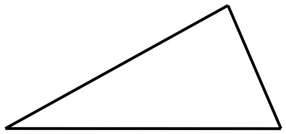
Triangles and Angles

Triangles are classified by the measure of their angles and sides

- Triangles classified by **angle**

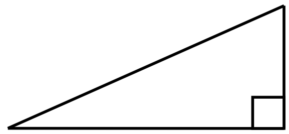
Examples:

Acute triangle



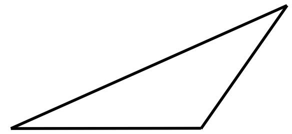
3 acute angles

Right triangle



1 right angle

Obtuse triangle

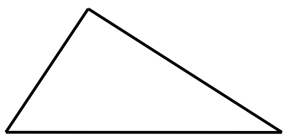


1 obtuse angle

- Triangles classified by **sides**

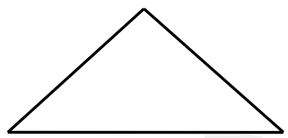
Examples:

Scalene triangle



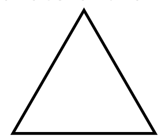
No equal sides

Isosceles triangle



2 equal sides

Equilateral triangle

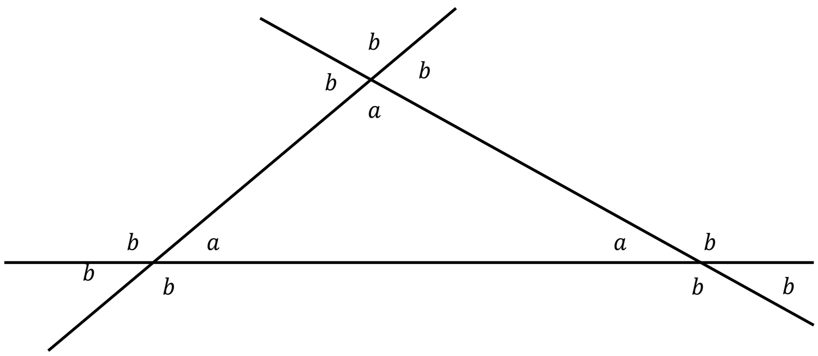


3 equal sides

Angles in triangles are related

- **Interior angles** are **angles** within/inside of a **polygon**
- **Exterior angles** are **angles** formed on the outside of a geometric shape. By extending one side past the **vertex**

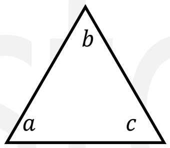
Example:



a = **Interior Angles**
b = **Exterior Angles**

- The **sum of the interior angles** of a triangle is 180°

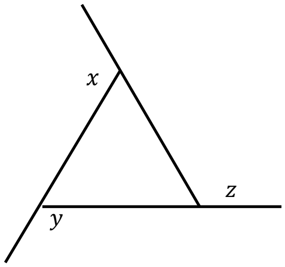
Example:



$a + b + c = 180°$

- The **sum of the exterior angles** of a triangle is 360°

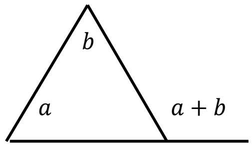
Example:



$x + y + z = 360°$

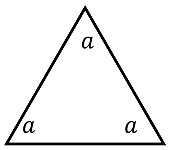
- The **exterior angle** at each **vertex** of a triangle is equal to the **sum** of the **interior angles** at the opposite **vertices**

Example:

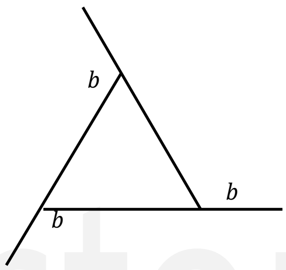


- **Equiangular** is a type of triangle that has all equal **angles**

Example:



$a + a + a = \frac{180°}{3} = 60°$



$b + b + b = \frac{360°}{3} = 120°$

Pythagorean Theorem

The **square** on the **hypotenuse** is equal to the **sum** of the **squares** on the other two **sides**

- Only applies to a **right angle** triangle
- The **hypotenuse** is always the longest **side** of the triangle
- To solve, work step by step to solve for the missing length

H
c
a

b

Formula:

$a^2 + \; b^2 = \; c^2$ or

$\sqrt{a^2 + b^2} = c$

c
5c

7c

Example:

$a^2 + b^2 = c^2 c^2 = a^2 + b^2 c^2 = \; 5^2 + 7^2$

$c^2 = 25 + 49$

$c = \sqrt{74}$

$c = 8.6\text{cm}$

- You can also check your work by confirming the rule that the **hypotenuse** is the longest **side** and/or you can plug in your value and eliminate a given length.

Example: $a^2 + b^2 = c^2$

$b^2 = c^2 - a^2 b^2 = \; 8.6^2 - 5^2$

$b^2 = \; 74 - 25$

$b = \sqrt{49}$

$b = 7\text{cm}$

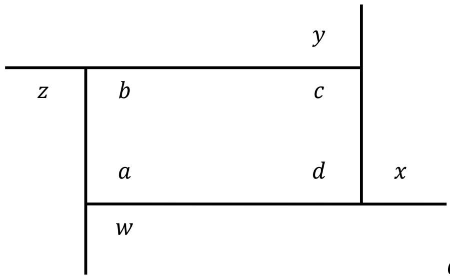
- A **Pythagorean** triple is when $a = 3; b = 4; c = 5$

Quadrilaterals and Angles

Angles in quadrilaterals are related

- The **sum of the interior angles** of a quadrilateral is 360°

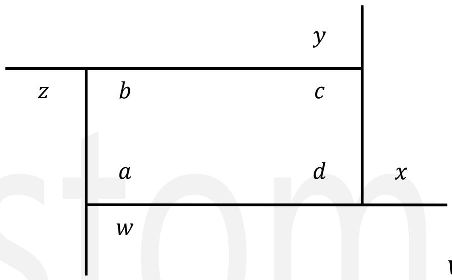
Example:



$a + b + c + d = 360°$

- The **sum of the exterior angles** of a quadrilateral is 360°

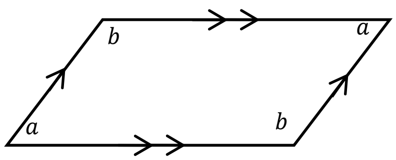
Example:



$w + x + y + z = 360°$

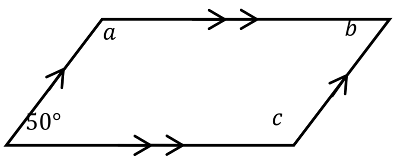
- **Angles** in parallelograms are related for **opposite angles** are equal

Example:



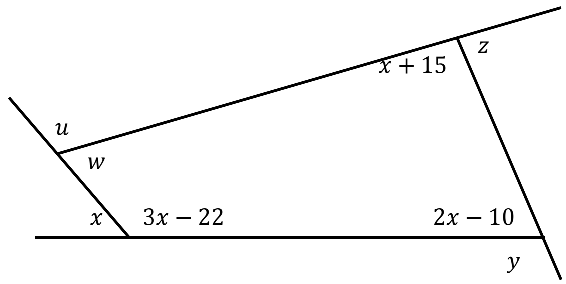
- When given 1 **angle** in a parallelogram, we can solve the rest of the angles using **supplementary angles**

Example:



$a = 180° - 50°$
$a = 130°$
$b = 50°$
$c = 130°$

- When given shapes with unassigned **variables**, it is possible to solve



Example:

Start with the **angle** with both **variables** and combine them to equal $180°$

$x+3x-22=180$

$4x-22+22=180+22$

$\frac{4x}{4}=\frac{202}{4}$

$x=50.5°$

Solve for the rest of the variables

$y=2x-10$

$y=2(50.5)-10$

$y=101-10$

$y=91; 180-91$

$y=89° z=x+15$

$z=50.5+15$

$z=65.5°$

Since we know that the **interior angles** of a quadrilateral **add** up to $360°$, combine the other values to find u and w

$360-(3x-22+2x-10+x+15)=w$

$360-(6x-17)=w$

$360-(6(50.5)-17)=w$

$360-(303-17)=w$

$360-286=w$

$w=74°$

$u=180-74$

$u=106°$

Polygons and Angle Relationships

Angles in and **polygon** are related

- The **sum of the interior angles** (**SIA**) of a **regular polygon** and its sides, n, can be **expressed** algebraically (refer to chart at the end of this section)

  Formula: $180(n-2); n = \text{number of } \textbf{sides}$

  Example: Octagon, find the **sum of interior angles** (**SIA**) $\therefore n = 8$

  $180(8-2) = 1080°$

  $\therefore$ the SIA of an octagon is $1080°$

- We can determine how many **sides** a **polygon** has when given its **SIA**

  Example:

  $\text{SIA} = 180$

  $\text{SIA} = 180(n-2)$

  $180 = 180°(n-2)$

  $180 = 180°(n) + 180(-2)$

  $180 = 180°n - 360$

  $180 + 360 = 180n - 360 + 360$

  $\frac{540}{180} = \frac{180n}{180}$

  $n = 3$

  $\therefore$ the polygon has 3 sides. It is a triangle

- We can determine EACH **angle** of the **regular polygon** when given its **SIA**

  Example:

  $\text{SIA} = 140n$

  $\text{SIA} = 180(n-2)$

  $140n = 180(n-2)$

  $140n = 180(n) + 180(-2)$

  $140n = 180n - 360$

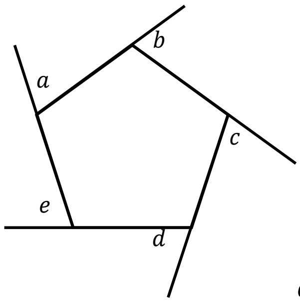
  $140n - 180n = -360$

  $\frac{-40n}{-40} = \frac{-360}{-40}$

  $n = 9$

  $\therefore$ The polygon with interiod angles of $140°$ has 9 **sides**

- The **sum of the exterior angles** of a **convex polygon** is $360°$

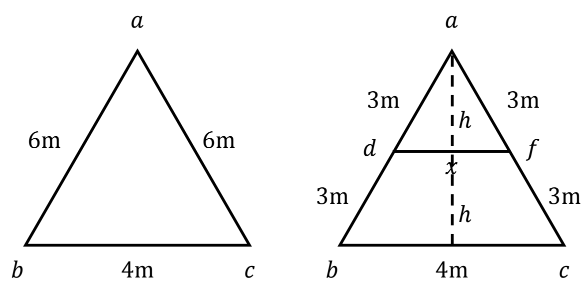
Example: 

$a + b + c + d + e = 360°$

Midpoints and Medians

Triangles and midpoints

- Triangles can be broken down into **midpoints** and **medians**
- The **midpoint** is the point that **divides** a **line** into 2 equal **segments**



Example:

$bd = da$, point d is the midpoint of side $abcf$

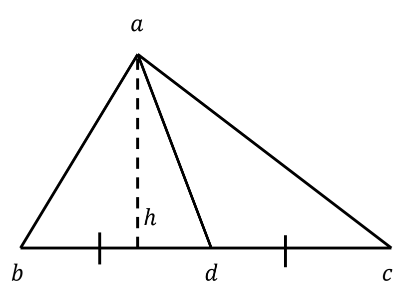
$= fa$ point f is the midpoint of side $acde = \frac{1}{2}\ bc$

$\because bc = 4\text{m}$ then $de = 2\text{m} \because d$ and e are midpoints of 2 sides of Δabc

The height of $\Delta ABC =$ height of trapezoid $decb$

Triangles and medians

- The **line segment** joining a **vertex** of a triangle to the **midpoint** of the opposite **side**
- A **bisect** is when a **line segment** cuts the **area** of a triangle in half by connecting **vertex** and the opposite **midpoint dividing** the **area** into 2 equal parts
- A **right bisector** is when a **line** is **perpendicular** to a **line segment** and passing through its **midpoint**

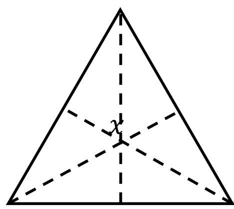


Example:

Area of $\Delta abd =$ area of $\Delta adc \because A = \frac{Bh}{2}; B = \text{base}; h$

$=$ HeightMedian cuts Δabc into 2 equal parts. It **bisects** the area

- A **centroid** is the **point** where the **medians** of a triangle intersect

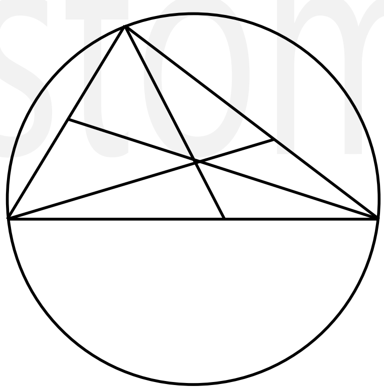
Example: 

$x = \text{centroid}$

- A **line segment** joining the **midpoints** of 2 sides of a triangle is **parallel** to the 3rd side and half as long
- The height of a triangle formed by joining **midpoints** of 2 sides of a triangle is half the height of the original triangle
- The **medians** of a triangle **bisect** its area
- A **diagonal** is a **line segment** joining 2 non-**adjacent vertices** of a **polygon**

Circumference of a triangle

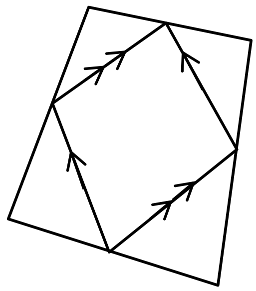
- The **circumference** of a **triangle** is the **point of intersection** of the **perpendicular bisectors** of the sides of the **triangle**
- This **point** serves as the **centroid** of a **circle** which passes through all of the **vertices** of a **triangle**

Example: 

Quadrilaterals and midpoints

- Joining the **midpoints** of the **sides** of any quadrilateral produces a **parallelogram**

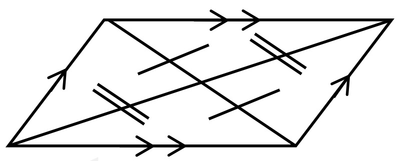
Example:



Quadrilaterals and medians

- The **diagonals** of a parallelogram **bisect** each other

Example:

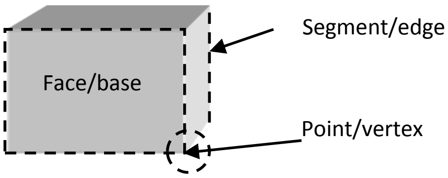


Geometric Figures

Polyhedron

- A **polyhedron** is a 3 dimensional **polygon**
- A **polyhedron** has a **face** called the **base**
- A **polyhedron** has **line segments** where 2 **faces** meet called an **edge**
- A **polyhedron** has points where **edges** meet called a **vertex** or **vertices**

Example:



Solids, shells and skeletons

- A **solid** is a 3 dimensional object whose interior is completely filled

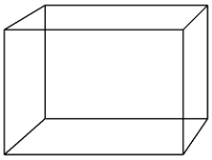
Example:

- A **shell** is a 3 dimensional object whose interior is completely hollow

Example:

- A **skeleton** is a representation of the **edges** of a **polyhedron**

Example:

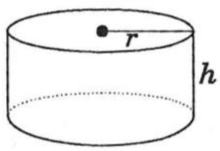
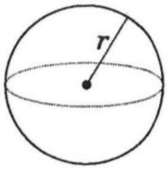
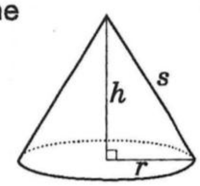
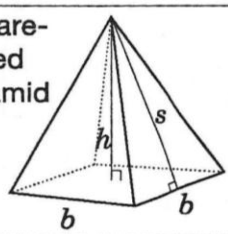
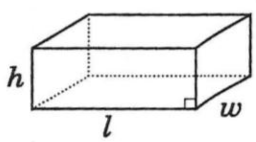
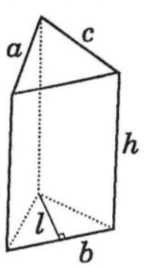


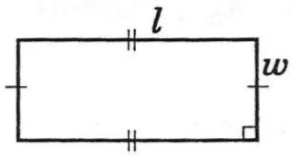
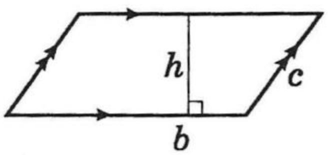
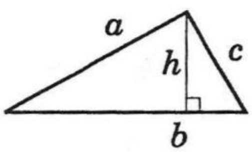
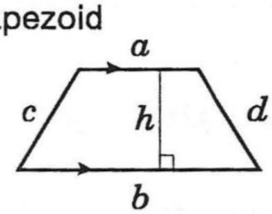
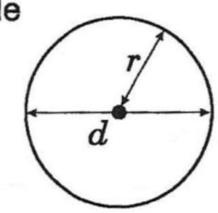
- **Surface Area** is the area on the outer shell

Formula: $A = A_3 + A_2 + A_3 \ldots$

- **Volume** is the amount of space an object takes up

Formula: $V = b \times h \times w$ or bwh

Geometric Figure	Area/Surface Area	Volume
Cylinder 	$A_{\text{base}} = \pi r^2$ $A_{\text{lateral surface}} = 2\pi r h$ $A_{\text{total}} = A_{\text{2 bases}} + A_{\text{lateral surface}}$ $= 2\pi r^2 + 2\pi r h$	$V = (A_{\text{base}})(\text{height})$ $V = \pi r^2 h$
Sphere 	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$ or $V = \frac{4\pi r^3}{3}$
Cone 	$A_{\text{lateral surface}} = \pi r s$ $A_{\text{base}} = \pi r^2$ $A_{\text{total}} = A_{\text{lateral surface}} + A_{\text{base}}$ $= \pi r s + \pi r^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3}\pi r^2 h$ or $V = \frac{\pi r^2 h}{3}$
Square-based pyramid 	$A_{\text{triangle}} = \frac{1}{2} b s$ $A_{\text{base}} = b^2$ $A_{\text{total}} = A_{\text{4 triangles}} + A_{\text{base}}$ $= 2bs + b^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3} b^2 h$ or $V = \frac{b^2 h}{3}$
Rectangular prism 	$A = 2(wh + lw + lh)$	$V = (\text{area of base})(\text{height})$ $V = lwh$
Triangular prism 	$A_{\text{base}} = \frac{1}{2} b l$ $A_{\text{rectangles}} = ah + bh + ch$ $A_{\text{total}} = A_{\text{rectangles}} + A_{\text{2 bases}}$ $= ah + bh + ch + bl$	$V = (A_{\text{base}})(\text{height})$ $V = \frac{1}{2} b l h$ or $V = \frac{blh}{2}$

<table>
<tr><th>Geometric Figure</th><th>Perimeter</th><th>Area/Surface Area</th></tr>
<tr><td>Rectangle

</td><td>$P = l + l + w + w$
or
$P = 2(l + w)$</td><td>$A = lw$</td></tr>
<tr><td>Parallelogram

</td><td>$P = b + b + c + c$
or
$P = 2(b + c)$</td><td>$A = bh$</td></tr>
<tr><td>Triangle

</td><td>$P = a + b + c$</td><td>$A = \frac{bh}{2}$
or
$A = \frac{1}{2}bh$</td></tr>
<tr><td>Trapezoid

</td><td>$P = a + b + c + d$</td><td>$A = \frac{(a + b)h}{2}$
or
$A = \frac{1}{2}(a + b)h$</td></tr>
<tr><td>Circle

</td><td>$C = \pi d$
or
$C = 2\pi r$</td><td>$A = \pi r^2$</td></tr>
</table>

- Frustum Pyramid

Formula: $\frac{1}{3}h(A_1 + A_2 + \sqrt{(A_1 \times A_2)})$

Optimization of Measurements

It is possible to find the maximum **area** with a given **perimeter** through **optimization**

- **Optimization** is the process of finding values that make a given quantity the greatest or least possible
- **Maximum** means the greatest possible
- **Optimizing** the **area** of a rectangle means finding the dimensions of the rectangle with maximum **area** for a given **perimeter**
- For a rectangle with a given **perimeter**, there are dimensions that result in the maximum **area**
- The dimensions of a rectangle with **optimal area** depend on the number of **sides**. If the **perimeter** is not required on all **sides**, a greater area can be enclosed

Formula:

4 **sides**

Length and Width $= \frac{P}{4}$

3 **sides**

Length $= \frac{P}{2}$

Width$= \frac{L}{2}$;

2 **sides**

Length and Width $= \frac{P}{2}$

Trigonometry

Relations between triangles

Used to find the relations between **angles** and **lengths** of **triangular** shapes

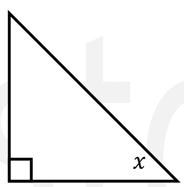
- Ensure that **radians** are being used
- **Congruent** means exact
- **Similar** means shared **angles** and some **side lengths**
- There are only 4 cases of congruency

Formula:	side, side, side	SSS
	side, angle, side	SAS
	angle, side, angle	ASA
	hypotenuse, side	HyS

Similar Triangles

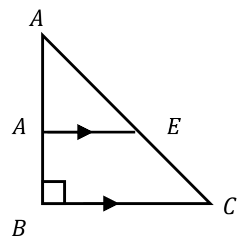
- One **triangle** is similar to another **triangle** if 2 out of the 3 **angles** are the same

Example:



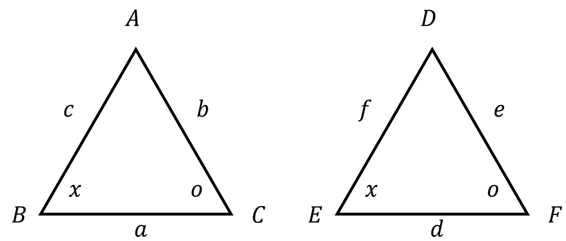
- Similar does not indicate that the lengths are equal but if **triangles** are similar, there are some **ratios** that result
- AA means **angle** to **angle** similarity

Example:



In ΔABC and ΔADE
$\angle ABC = \angle ADE$ Authority: Parallel lines (F pattern)
$\angle ACB = \angle AED$ Authority: Parallel Lines (F pattern)
$\angle BAC = \angle DAE$ Authority: Common
$\therefore \Delta ABC \sim \Delta ADE$ Authority: AA~

- Shared similarities

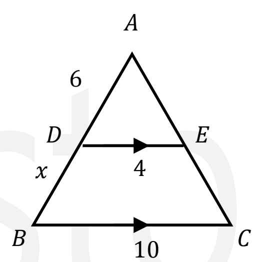


Example:

If $\Delta ABC \sim \Delta DEF$

1. $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$
2. $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$
3. $\frac{\text{Area}\Delta ABC}{\text{Area}\Delta DEF} = \frac{a^2}{d^2} = \frac{b^2}{e^2} = \frac{c^2}{f^2}$

- Proofing similarities



Example:

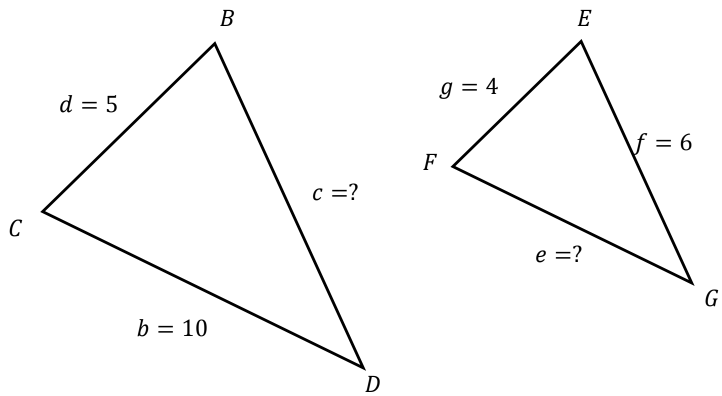
In ΔABC and ΔADE
$\angle ABC = \angle ADE$ (**corresponding** – F pattern)
$\angle ACB = \angle AED$ (**Corresponding** – F pattern)
$\therefore \Delta ABC \sim \Delta ADE$ By the AA similar **triangle** theorem

- Cross **multiply**

In ΔABC and ΔADE

$$\frac{6}{4} = \frac{6+x}{10}$$
$$6 \times 10 = 4(6+x)$$
$$60 = 24 + 4x$$
$$60 - 24 = 4x$$
$$36 = 4x$$
$$\frac{36}{4} = x$$
$$x = 9$$

- **Ratios** of similar **triangles** and **lengths**



Example:

In ΔABC and ΔEFG

$\angle CBD = \angle FEG$ given

$\angle DCB = \angle GFE$ given

$\therefore \Delta ABC \sim \Delta EFG$ AA~

$\frac{5}{4} = \frac{c}{6}$

$\frac{5 \times 6}{4} = c$

$\frac{30}{4} = c$

$7.5 = x$

$\frac{b}{e} = \frac{d}{g}$

$\frac{10}{e} = \frac{5}{4}$

$10 \times 4 = 5 \times e$

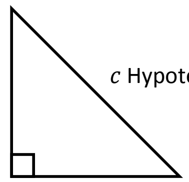
$40 = 5 \times e$

$\frac{40}{5} = e$

$8 = \text{e}$

Trigonometry Laws

- Applies to a **right angle triangle**
- Always use **degrees** for calculations
- A **trigonometric ratio** is a **ratio** of the **length** of 2 sides in a **right angled triangle**
- Standard **triangle** layout



Formula:

- A **theta** is the **angle** to be found or given

Formula: $\Theta = \text{theta}$

- The **opposite** is the **side** not joined by the **vertex** where the **theta** is. The **adjacent** is the **side** next to the **opposite**
- These laws define how to find the **theta** is any position within the **triangle**

Formula: $\text{SOH} - \text{CAH} - \text{TOA}$

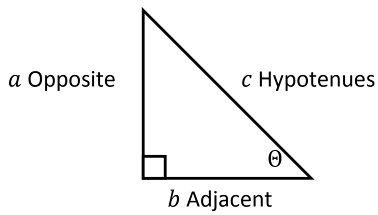
$$\tan\Theta = \frac{\text{opp}}{\text{adj}}; \text{Tangent}$$

$$\sin\Theta = \frac{\text{opp}}{\text{hyp}}\ ; \text{Sine} \cos\Theta = \frac{\text{adj}}{\text{hyp}} \text{Cosine}$$

- **Angle of elevation** is from base line and up
- **Angle of depression** is from top and down
- **Round** to 3 **decimal places**

The tangent ratio

- When given the **opposite** and/or **adjacent** and/or **theta**, 2 of these values when used with the **tangent ratio** will result in finding the missing value

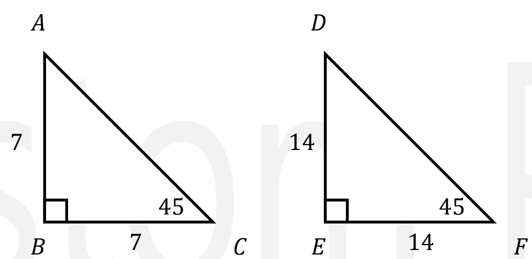


Formula:

- A 45° angle when placed with the **tangent ratio** will result in 1

Example: $\tan 45\,° = \frac{7}{7} = 1 \leftarrow \text{from } \Delta\, ABC$

$$\tan\ 45° = \frac{14}{14} = 1 \leftarrow \text{from } \Delta\, DEF$$



- To solve, plug in values into their placeholders

Example: A person is standing 27m from the **base** of a tree. The **angle of elevation** to the top is 57°. Find the trees **height**

$$\tan \Theta = \frac{opp}{adj}$$
$$\tan 57 = \frac{h}{27}$$
$$27(\tan(57)) = h$$
$$27 \tan 57° = h$$
$$h = 41.576\text{m}$$

- Using the **arctan** or the negative of **tan** will find the **theta**

Example: $\tan\theta$; $\arctan\theta$ or $\tan^{-1}\theta$

Example: A sniper is at the top of a 108m tall building, aiming at a cat that is 81m from the front door of the building. At what **angle of declination** must the sniper aim at to get the cat?

$\tan\Theta = \frac{opp}{adj}$

$\tan\Theta = \frac{81}{108}$ $\Theta = tan^{-1}\left(\frac{81}{108}\right)$ $\Theta = 36.86 = 37°90° - \Theta = 53°z$

∴ the angle of depression is 53°

Example: Solve for x

$\tan 13 = \frac{71}{x}$

Solution 1

$x\frac{(\tan 13)}{1} = x\left(\frac{71}{x}\right)x(\tan 13) = 71$

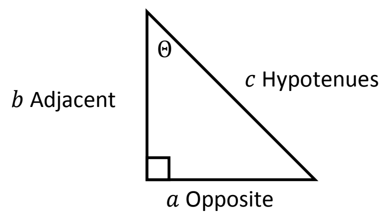
$x = \frac{71}{\tan 13}$

Solution 2

$\left(\frac{(\tan 13)}{1}\right)^{-1} = \left(\frac{71}{x}\right)^{-1}\frac{1}{\tan 13} = \frac{x}{71}\frac{71}{\tan 13} = x$

The sine ratio

- The **ratio** does not depend on the size of the **triangle**, only the size of the **angle**
- When given the **opposite** and/or **hypotenuse** and/or **theta**, 2 of these values when used with the **sine ratio** will result in finding the missing value



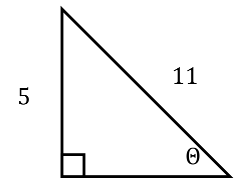
Formula:

Example: You are looking at the top of the math tower. A statue of Pythagoras is at the top. You are looking up at an angle of 62° and through use of GPS; you know Pythag is 60m from you.

$$\sin 62 = \frac{x}{60}$$
$$60 \sin 62 = x$$

$$x = 52.97$$

Example: Solve this **triangle**



$\sin\theta = \frac{5}{11}$
$\theta = 27$
$\angle B = 90 - 27$
$\angle B = 63$

Side AC can be found in 3 ways

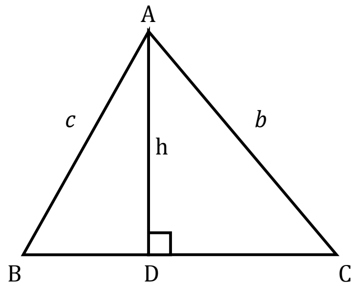
Pythagorean Theorem
$11^2 - 5^2 = b^2$
$9.8 = b$

Trigonometry
$\tan 27 = \frac{5}{b}$
$b = \frac{5}{\tan 27}$
$b = 9.8$

Trigonometry
$\sin 63 = \frac{b}{11}$
$11 \sin 3 = b$
$b = 9.8$

The sin law

- The **sine** law allows you to perform calculations on **triangle** that are not **right angled**



Example:

In ΔABD

$$\sin B = \frac{h}{c}$$

$$h = c \sin B$$

$$In\ \Delta ACD$$

$$\sin C = \frac{h}{b}$$

$$h = b \sin C$$

$$h = h$$

$$c \sin B = b \sin C$$

$$\frac{c \sin B}{b} = \sin C$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example:

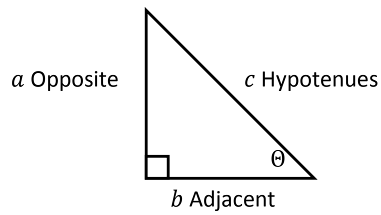
$$\frac{80}{\sin 66} = \frac{x}{\sin 63}$$

$$\frac{80 \sin 63}{\sin 66} = x$$

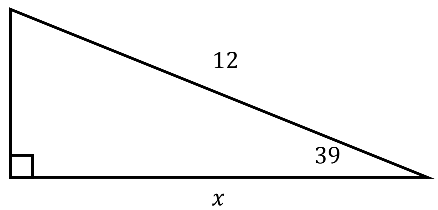
$$x = 78$$

The cosine ratio

- When given the **adjacent** and/or **hypotenuse** and/or **theta**, 2 of these values when used with the **cosine ratio** will result in finding the missing value



Formula:



Example: find x

$$\cos 39 = \frac{x}{12}$$
$$12\cos 39 = x$$
$$x = 9.33$$

The cosine law

- **Dividing sine** and **cosine**

Formula:

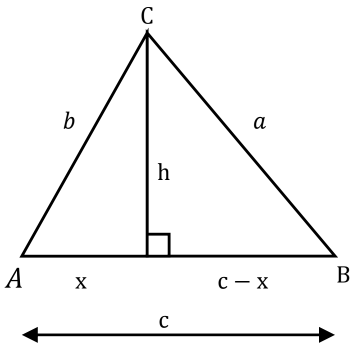
$$\text{ratio} = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\frac{opp}{adj}}{\frac{adj}{hyp}}$$
$$= \frac{opp}{hyp} \times \frac{hyp}{adj}$$
$$= \frac{opp}{adj}$$
$$= \tan\theta$$

Example:

$$\sin^2\theta - \sin\theta = 0$$
$$(\sin\theta)^2 - \sin\theta = 0$$
$$\sin\theta(\sin\theta - 1) - 0$$
$$\sin\theta = 0; \theta = 0$$
$$\sin\theta = 1; \theta = 90$$
$$(\sin 90)^2 - \sin 90 = 0$$
$$(\sin 0)^2 - \sin 0 = 0$$

Example:

$$2(\cos x)^2 - 7\cos x + 3 = 0$$
Compared to
$$2x^2 - 7x + 3 = 0$$
$$2(\cos x)^2 - 6\cos x - \cos\ x + 3 = 0$$
$$2\cos x\,(\cos x - 3) - 1(\cos x - 3) = 0$$
$$(2\cos x - 1)(\cos x - 3) = 0$$
$$2\cos x - 1 = 0; \cos x = \frac{1}{2}; x = 60$$
$$\cos x - 3 = 0; \cos x = 3; x = \text{inadmissable, rejected}$$



Example:

In ΔABC, draw CD perpendicular to AB, CD is the altitude, h, of ΔABC

Let $AD = x$

In ΔACD $b^2 = h^2 + x^2$

$BD = c - x$

$\frac{x}{b} = \cos A$

$x = b\cos A$

In ΔBCD $a^2 = h^2 + (c-x)^2 (\text{pythag})$

$a^2 = h^2 + c^2 - 2cx + x^2$

$a^2 = h^2 + x^2 + c^2 - 2cx$

but $b^2 = h^2 + x^2$ and $x = b\cos A$

$a^2 = c^2 + b^2 - 2cb\cos A$

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

$a^2 - b^2 - c^2 = -2bc\cos A$

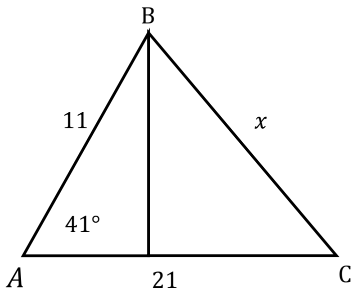
$\frac{(a^2 - b^2 - c^2)}{-2bc} = \cos A$

$-\frac{a^2 - b^2 - c^2}{2bc} = \cos A$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



Example:

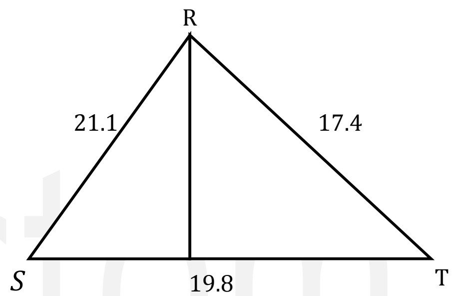
Find x

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 21^2 + 11^2 - 2(21)(11)\cos 41$

$a^2 = 213.32$

$a = 14.6$



Example:

Find $\angle S, \angle R, \angle T$

$\cos S = \frac{((12.1)^2 + (19.8)^2 - (17.4)^2)}{2(12.1)(19.8)}$

$\cos S = 0.491882626$

$\angle S = 60.5$

$\angle = 61$

$\cos R = \frac{(12.1)^2 + (17.4)^2 - (19.8)^2}{2(2.1)(17.4)}$

$\angle R = 82$

$\angle T = 180 - 61 - 82$

$\angle T = 37$

Formula:

$a^2 = b^2 + c^2 - 2bc\cos a$ or $\cos a = \frac{b^2+c^2-a^2}{2bc}$

$b^2 = a^2 + c^2 - 2ac\cos b$ or $\cos b = \frac{a^2+c^2-b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab\cos c$ or $\cos c = \frac{a^2+b^2-c^2}{2ab}$

- **Graphing quadrants** or **casts** have **trigonometry** relationships
- There are 4 **quadrants** when **graphing**. The **quadrants** are labelled in counter clockwise order starting at the top right **quadrant**. **Quadrants** are also known as the **cast**

2 sin+	**1** all+
3 tan+	**4** cos+

- **Trigonometry ratios** fall on certain **quadrents**

Given: $\cos\pm;\ \sin\pm;\ \tan\pm$

Qudrant 1: All **ratios** are positive;

Qudrant 2: **Sine ratio** is positive

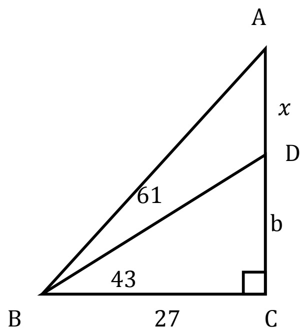
Qudrant 3: **Tangent ratio** is positive

Qudrant 4: **Cosine ratio** is positive

Working with 2 right triangles

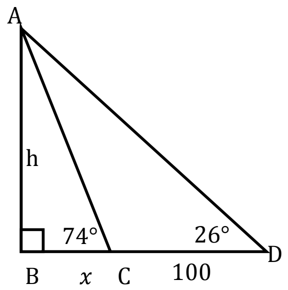
- When given enough information, anything about a **triangle** can be found through use of **tragicomic ratios**

Example:



find x

$$\tan 43 = \frac{b}{27}$$
$$27\tan 43 = b$$
$$b = 25.18$$
$$\tan 61 = \frac{x+b}{27}$$
$$\tan 61 = \frac{x + 27\tan 43}{27}$$
$$27\tan 61 = x + 27\tan 43$$
$$27\tan 61 - 27\tan 43 = x$$
$$x = 23.53$$



Example:

In ΔABC

$$\tan 74 = \frac{h}{x}$$

$$x = \frac{h}{\tan 74}$$

In ΔABD

$$\tan 26 = \frac{h}{100 + x}$$

$$100 + x = \frac{h}{\tan 26}$$

$$x = \frac{h}{\tan 26} - 100$$

$$x = x$$

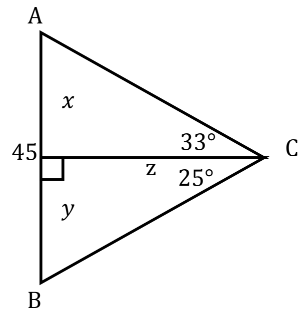
$$\frac{h}{\tan 74} = \frac{h}{\tan 26} - 100$$

$$100 = \frac{h}{\tan 26} - \frac{h}{\tan 74}$$

$$100 = h\left(\frac{1}{\tan 26} - \frac{1}{\tan 74}\right)$$

$$\frac{100}{\frac{1}{\tan 26} - \frac{1}{\tan 74}} = h$$

$$h = 5.67$$



Example: Find z

In ΔADC

$\tan 33 = \frac{x}{z}$

$z \tan 33 = x$

In ΔBCD

$\tan 25 = \frac{y}{z}$

$x + y = 45$

$z \tan 33 + z \tan 25 = 45$

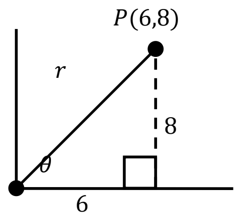
$z(\tan 33 + \tan 25) = 45$

$z = \frac{45}{(\tan 33 + \tan 25)}$

$z = 40.33$

Trigonometric ratios greater than right angles

- When given a **point** on a **graph**, several items can be determined through **trigonometry** based on relations
- The relations always begin in the first **quadrant** (upper-left), this is known as **standard position**
- The x axis is known as the **initial arm**, or where the **theta** begins
- The **terminal arm** defines the size of the **angle**
- The **vertex** is on the **origin**
- From the **point**, we can determine the **hypotenuse** through use of the **Pythagorean theorem**, then solve the **theta** through use of $SOH - CAH - TOA$



Example:

$$r = \sqrt{6^2 + 8^2}$$
$$r = 10$$
$$\sin\theta = \frac{8}{10}$$
$$\cos\theta = \frac{6}{10}$$
$$\tan\theta = \frac{8}{6}$$

Formula:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

- When there is an **obtuse angle**, there can be 2 possibilities depending on the ratio used

Example: $P(-4,3)$
$r = 5$
$\sin\theta = \frac{3}{5} \to 37$
$\cos\theta = -\frac{4}{5} \to 143$
$tan\theta = -\frac{3}{4} \to -37$

Formula: $\sin\theta = \sin(180 - \theta)$
$\cos\theta = -\cos(180 - \theta)$
$\tan\theta = -\tan(180 - \theta)$

- When given **restrictions**, the number of possibilities can be reduced

Example: $90° \leq \theta \leq 180° \,|\, \sin\theta = 0.8191 \to 125°$
$90° \leq \theta \leq 180° \,|\, \cos\theta = -0.7431 \to 138°$
$0° \leq \theta \leq 180° \,|\, \sin\theta = 0.9903 \to 82°, 98°$
$0° \leq \theta \leq 180° \,|\, \cos\theta = 0.9205 \to 23°$

- It is possible to have a negative **terminal arm**
- When the arm **rotates** against 0°, the **angle** becomes negative
- In a case where a positive **terminal** arm is greater than 360°, it indicates more than 1 revolution, therefore it can be equal to another positive **terminal arm**, this is called **co-terminal angles**

Example: $420°$ and $60°$ are co-terminal

The sine law: ambiguous case

- When 2 **sides** and the non-included **angle** of a **triangle** are given, the **triangle** may be unique
- With this info, there are 3 cases: there is no **triangle**, 1 **triangle**, or 2 **triangles** based on the measurements
- Cases for when the angle $A < 90°$; you will be given **sides** a and b

Example: If $a \geq b$ then there is 1 exact solution

Example: If $a < b$ and,
$a = b \sin A$ and,
$\angle B = 90°$ then there is 1 exact solution

Example: if $a < b$ and,
$a < b \sin A$ then there is no exact solution

Example: If $a < b$ and,
$a > b \sin A$ and,
Sine ratios of **supplementary angles** are equal
then there is 2 exact solutions; 1 **acute triangle**, 1 **obtuse triangle**

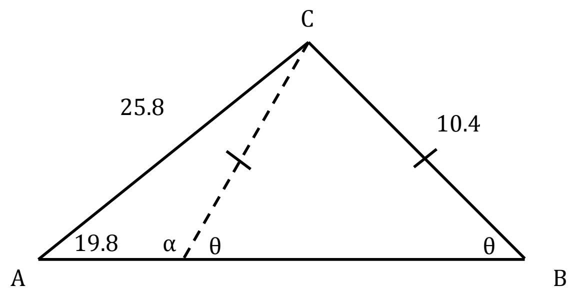
- Cases for when the angle $A > 90°$; you will be given **sides** a and b

Example: If $a \leq b$ then there is no exact solution

Example: If $a > b$ then there is 1 exact solution

- The **ambiguous case** is really just a matter of determining how many solutions are available when you use the **sine law**

Example: Solve ΔABC if $\angle A = 19.8°, b = 25.8, a = 10.4$



$$\frac{\sin 19.8}{10.4} = \frac{\sin B}{25.8}$$
$$\angle B = \sin^{-1}\left(\frac{25.8 \sin 19.8}{10.4}\right)$$
$$\angle B = 10.4$$
$$\alpha = 180 - 57.2$$
$$\alpha = 122°$$

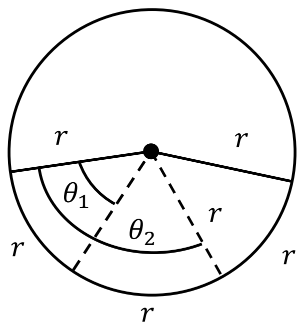
$$\Delta ABC \text{ (acute)} \therefore \angle B = 57.2°$$
$$\angle C = 180 - 76.4$$
$$\angle C = 103°$$
$$\frac{c}{\sin 103} = \frac{10.4}{\sin 19.8}$$
$$c = \frac{10.4 \sin 103}{\sin 19.8}$$
$$c = 29.9$$

$$\Delta ABC \text{ (obtuse)} \therefore \angle B = 122.8°$$
$$\angle C = 180 - 142.6$$
$$\angle C = 37.4°$$
$$\frac{c}{\sin 37.4} = \frac{10.4}{\sin 19.8}$$
$$c = \frac{10.4 \sin 37.4}{\sin 19.8}$$
$$c = 18.6$$

Radians and Angle Measure

- Angles can be measured in **radians** and converted from **degrees**
- One **radian** is the measure of the **angle** subtended at the centre of a **circle** by an **arc** that is equal in length to the **radius** of the **circle**

Formula: $\theta_1 = 1$ radian
$\theta_2 = 2$ radian
$\theta_3 = 3$ radian



If $\theta_1 = 1$
it can be written as a ratio of $\frac{r}{r} = \theta_1$
$\theta_2 = 2 = \frac{2r}{r}$
$\theta_3 = 3 = \frac{3r}{r} \therefore \text{(in radians)}\theta = \frac{\text{arc\_length}}{r} \rightarrow \theta = \frac{a}{r}$

- In order to compare **radians** to **degrees**, we must understand the length of the **arc**

Formula: $360°$ in a circle
$\theta = \frac{\text{arc\_length}}{r} = \frac{2\pi r}{r}$
$\therefore 360° = 2\pi$ radians
$180° = \pi$ radians
$\therefore 1° = \frac{\pi}{180°}$ rad or 1 rad $= \frac{180°}{pi}$

- Find the equivalent **radian** measure for degrees by using the formula
- Remember to reduce

Example: $50°$

$50\left(\frac{\pi}{180}\right)$

$\frac{5\pi}{18} = 0.087$

Example: $210°$

$210\left(\frac{\pi}{180}\right)$

$\frac{7\pi}{6} = 3.6651$

Example: $\frac{3\pi}{4}$

$\frac{3\pi}{4}\left(\frac{180}{\pi}\right) = 135°$

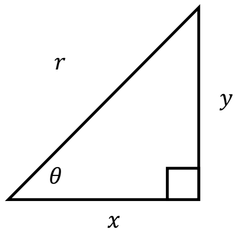
Example: 4.7

$4.7\left(\frac{180}{\pi}\right) = 269.2901$

Trigonometric ratios of any angle

- Keep **quadrants** in mind and as to when what **ratio** is positive in what **quadrant**. Refer to the acronym **CAST** (counter-clockwise from **quadrant** 4)

Formula:
$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$



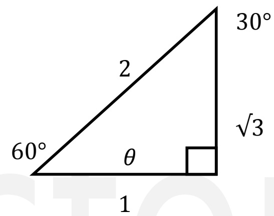
- The **Pythagorean Theorem** shows that the side lengths of a $30°|60°|90°$ **triangle** has the **ratio** of $1:\sqrt{3}:2$
- The side lengths of a $45°|45°|90°$ **triangle** has the **ratio** of $1:1:\sqrt{2}$

- These special cases of **triangles** have exact **ratios**

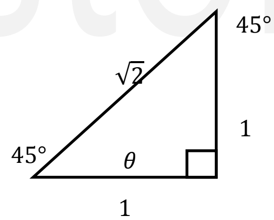
Formula:

θ in Degrees	θ in Radians	$sin\ \theta$	$cos\ \theta$	$tan\ \theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$0°$	0	0	1	0
$90°$	$\frac{\pi}{2}$	1	0	Undefined

Formula: 30°|60°|90°

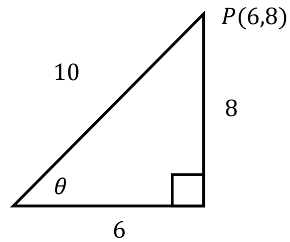


Formula: 45°|45°|90°



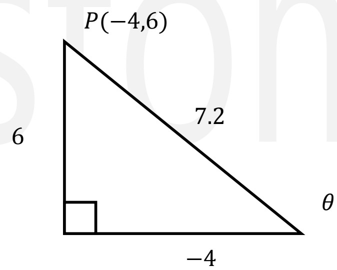
- Using the **ratios**, you can make definite of unknown measures

Example: $P(6,8)$



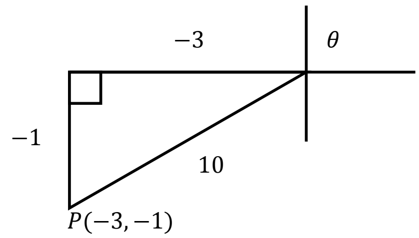
$r = \sqrt{8^2 + 6^2}$
$r = 10$
$\sin\theta = \frac{8}{10} \to \frac{4}{5}$
$\cos\theta = \frac{6}{10} \to \frac{3}{5}$
$\tan\theta = \frac{8}{6} \to \frac{4}{3}$
$\theta = 53.1°$

Example: $P(-4,6)$



$r = \sqrt{(-4)^2 + 6^2}$
$r = 7.2$
$\sin\theta = \frac{6}{7.2}$
$\cos\theta = -\frac{4}{7.2}$
$\tan\theta = -\frac{6}{4}$
$\theta = 123.6°$

Example: $P(-3,-1)$



$r = \sqrt{(-3)^2 + (-1)^2}$

$r = \sqrt{10}$

$\sin\theta = -\frac{1}{\sqrt{10}}$

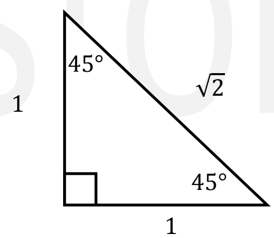
$\cos\theta = -\frac{3}{\sqrt{10}}$

$\tan\theta = \frac{1}{3}$

$\theta = 198.4°$

- When a question is asking for an exact result, it is looking for **radicals**, an approximate result would be looking for an answer in **radians**

Example: $\sin 135\,°$

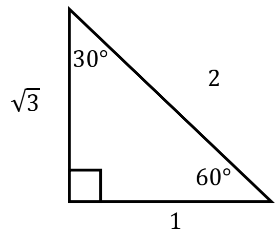


$180° - 135° = 45°$

$\sin 45° = \frac{1}{\sqrt{2}}$

$\sin 135° = \frac{1}{\sqrt{2}}$

Example: $\cos 120\,°$

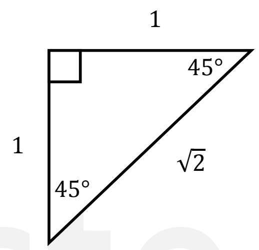


$180° - 120° = 60°$

$\cos 60° = \frac{1}{2}$

$\cos 120° = -\frac{1}{2}$

Example: $\tan 225°$



$270° - 225° = 45°$

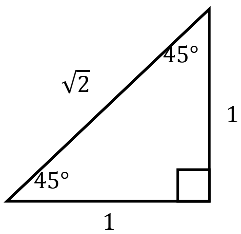
$\tan 45° = 1$

$\tan 225° = 1$

- Working with a **radian** value, remember the formula

Example: $\sin\frac{\pi}{4}$

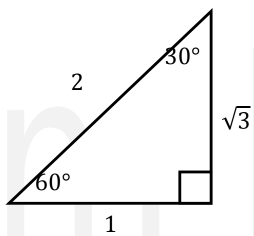
$\sin\frac{\pi}{4}\left(\frac{180}{\pi}\right)\sin 45°$



$\sin 45° = \frac{1}{\sqrt{2}}$ or $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Example: $\cos\frac{\pi}{6}$

$\cos\frac{\pi}{6}\left(\frac{180}{\pi}\right)\cos 30°$

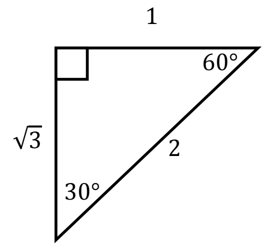


$\cos 30° = \frac{\sqrt{3}}{2}$ or $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

- Always check to see if there are 2 possibilities and whether they are negative or positive

Example: $\sin\frac{4\pi}{3}$

$\sin\frac{4\pi}{3}\left(\frac{180}{\pi}\right)\sin 240°$

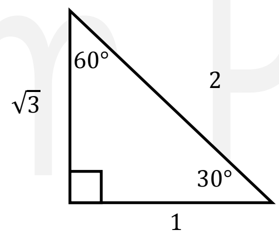


$240° - 180° = 60° \sin 60° = \frac{\sqrt{3}}{2}$

$\sin 240° = -\frac{\sqrt{3}}{2}$

Example: $\cos\frac{5\pi}{6}$

$\cos\frac{5\pi}{6}\left(\frac{180}{\pi}\right)\cos 150°$

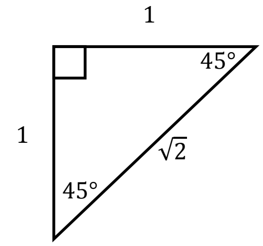


$180° - 150° = 30° \cos 30° = \frac{\sqrt{3}}{2}$

$\cos 150° = -\frac{\sqrt{3}}{2}$

Example: $\tan\frac{5\pi}{4}$

$\tan\frac{5\pi}{4}\left(\frac{180}{\pi}\right)\tan 225°$



$225° - 180° = 45° \tan 45° = 1$

$\tan 225° = 1$

- **Theta** in **standard position** $0 \leq \theta \leq 2\pi$ can solve the exact value of 2 **ratios**

Example: $\sin\theta = \frac{2}{5}\left[\frac{y}{r}\right]$

$x = \sqrt{5^2 - 2^2}$

$x = \pm\sqrt{21}$

Quadrant 1

$\cos\theta = \frac{\sqrt{21}}{5}$

$\tan\theta = \frac{2}{\sqrt{21}}$

Quadrant 2

$\cos\theta = -\frac{\sqrt{21}}{5}$

$\tan\theta = -\frac{2}{\sqrt{21}}$

Example: $\cos\theta = -\frac{1}{5}\left[\frac{x}{r}\right]$

$y = \sqrt{(-5)^2 - 1^2}$

$y = \pm\sqrt{24}$

Quadrant 2

$\sin\theta = \frac{\sqrt{24}}{5}$

$\tan\theta = -\frac{\sqrt{24}}{1} \to -2\sqrt{6}$

Quadrant 3

$\sin\theta = -\frac{\sqrt{24}}{5}$

$\tan\theta = \frac{\sqrt{24}}{1} \to 2\sqrt{6}$

Example: $\tan\theta = \frac{3}{7}\left[\frac{y}{x}\right]$

$r = \sqrt{3^2 + 7^2}$

$r = \pm\sqrt{58}$

Quadrant 1

$\sin\theta = \frac{3}{\sqrt{58}}$

$\cos\theta = \frac{7}{\sqrt{58}}$

Quadrant 3

$\sin\theta = -\frac{3}{\sqrt{58}}$

$\cos\theta = -\frac{7}{\sqrt{58}}$

Modelling periodic behaviour

- A **function** is **periodic** if it has a **pattern** of y values that repeats is at regular intervals
- A complete **pattern** is called a **cycle**. **Cycles** may begin at any **point** on a **graph**
- The **horizontal** length of a **cycle** is the **period** of the **function**

Example: (Involves a set of **ordered pairs**)
$(0,4), (8,4)$
$\because y_1 = y_2$
$\therefore$ the **period** is 8 units

- When given $f(x)$ and a **period**, it is possible to determine an x value at a specific time

Example: (Involves a set of **ordered pairs**)
$f(6) = -1$ (From looking at **graph**)
$f(20)$
$\because$ **period** is 7
$f(6) = f(6 + 7)$
$f(6) = f(6 + 7 + 7)$
$\therefore f(20) = -1$

- A **function** f is **periodic** if there exists a positive number p

Formula: $f(x + p) = f(x)$

- The smallest positive value of p is the **period** of the **function**
- The **amplitude** of the **function** is half the **difference** between the **max** and **min** of the **function**

Example: $\text{Max} = 3, \text{Min} = -1$
$$\text{Amplitude} = \frac{1}{2}(3 - (-1))$$
$\therefore$ the **amplitude** is 2

- State the **domain** and **range** is based on the **variables** used

Example: $D: \{0 \le t \le 12\}$
$R: \{0 \le d \le 800\}$

Transformations of trigonometric ratios

- **Transformations** that apply to **algebraic** expressions can also apply to **functions**
- If $a > 1$ then $y = a\sin x$ and $y = a\cos x$ are stretched vertically by a **factor** of a
- If $0 < a < 1$ then $y = a\sin x$ and $y = a\cos x$ are compressed vertically by a **factor** of a
- If $a < 0$ then there is a reflection on the x axis.
- a represents the **altitude** of the **function**

Example: $y = 3\sin x$

$y = \sin x$

$y = \frac{1}{3}\sin x$

- In one **cycle** of a **sine** or **cosine function**, there are 5 identifying points
- x intercept points are considered zero's, the **maximum** and **minimum** of the **function** is determined by the **altitude**, **corresponding** to a negative value is below 0

Example: $y = 3\sin x$, period $= 2\pi$

5 key points $= (0,0), \left(\frac{\pi}{2}, 3\right), (\pi, 0), \left(\frac{3\pi}{2}, -3\right), (2\pi, 0)$

- **Cosine function** begins at the **altitude** rather than 0

Example: $y = 4\cos x$, $(0,0), x \geq 0$

$(0,4), \left(\frac{\pi}{2}, 0\right), (\pi, 0), \left(\frac{3\pi}{2}, 0\right), (2\pi, 4)$

$D: 0 \leq x \leq 2\pi$

$R: -4 \leq y \leq 4$

- If $k > 1$ then $y = \sin kx$ and $y = \cos kx$ are compressed horizontally by a **factor** of $\frac{1}{k}$
- If $0 < k < 1$ then $y = \sin kx$ and $y = \cos kx$ are stretched horizontally by a **factor** of k
- 360° divided by the **period** results in horizontal **expansion** or **compression**
- The **period** will be either $\frac{2\pi}{k}$ or $\frac{360°}{k}$

Example: $y = \sin 3x$, period $= \frac{2\pi}{3}$

$(0,0), \left(\frac{\pi}{6}, 1\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{\pi}{2}, -1\right), \left(\frac{2\pi}{3}, 0\right)$

$D: 0 \leq x \leq \frac{2\pi}{3}$

$R: -1 \leq y \leq 1$

- When combining **transformations**, use the 5 point system to understand how to **graph** a **cycle**

Example: $y = 3\cos 2x$, domain $= -\pi \leq x \leq \pi$

Amplitude $= 3$

Max $= 3$

Min $= -3$

Period $= \frac{2\pi}{2} \rightarrow \pi$

$(0,3), \left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{2}, -3\right), \left(\frac{3\pi}{4}, 0\right), (\pi, 3)$

- When given a **graph** and a **coordinate**, plug in the values and solve

Example: $x = 670$, amplitude $= 1$

Period $= \frac{2\pi}{k}$

$670 = \frac{2\pi}{k}$

$k = \frac{2\pi}{670} \rightarrow \frac{\pi}{335}$

$\therefore y = \sin\left(\frac{\pi x}{335}\right)$

Approximate $= \frac{\pi}{335} \rightarrow 0.009$

$\therefore y = \sin 0.009x$

- For $y = a\sin x$ and $y = a\cos x$, the **amplitude** is $a (a > 0)$
- For $y = \sin kx$ and $y = \cos kx$, the **period** is $\frac{360}{k} (k > 0)$

Radians and Degrees

Radians ($\boldsymbol{x}$)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{11}$	$\frac{11\pi}{6}$	2π
Degrees ($\boldsymbol{x}$)	0	30	60	90	120	150	180	210	240	270	300	330	360

Translations of trigonometric ratios

- **Translations** that apply to **algebraic** expressions can also apply to **functions**
- If $c > 1$ then $y = \sin x + c$ and $y = \cos x + c$ are **translated** upward by c units
- If $c < 1$ then $y = \sin x + c$ and $y = \cos x + c$ are **translated** downward by c units
- Proper sequence of combinations is expansions and compressions, **reflections**, **translations**

Example: $y = 2\sin x + 3$
$\text{Amplitude} = 2$
$\text{Vertical Stretch} = 2$
$\text{Translate up} = 3$
$\text{Period} = 2\pi$
$D: 0 \le x \le 2\pi$
$R: 1 \le y \le 5$

- If $d > 0$ then $y = \sin(x - d)$ and $y = \cos(x - d)$ are **translated** right by $d$units
- If $d < 0$ then $y = \sin(x - d)$ and $y = \cos(x - d)$ are **translated** left by d units
- Horizontal **translations** are recognized as **phase shift** or **phase angle**

Example: $y = 0.5\cos\left(x + \frac{\pi}{2}\right)$
$\text{Amplitude} = 0.5$
$\text{Vertical compressed} = 0.5$
$\text{Phase shift left} = \frac{\pi}{2}$
$\text{Period} = 2\pi$
$D: -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$
$R: -0.5 \le y \le 0.5$

- When you put both **translations** and **transformations** for **functions**, follow the formula

Formula: $y = a\sin k(x-d)+c$

$y = a\cos k(x-d)+c$

Example: $y = 4\cos\left(\frac{1}{2}x+\frac{\pi}{2}\right)-1\,, -4\pi \le x \le 4\pi$

$y = 4\cos\frac{1}{2}(x+\pi)-1$

$\text{Period} = 4\pi$

Vertical Expansion: 4

Horizontal Expansion: 2

Amplitude: 4

$D = \{-\pi \le x \le 3\pi\}$

$R = \{-5 \le y \le 3\}$

Trigonometric identities

- An **identity** is an equation that is true for all values of the **variable** on both the left and rights sides of the equation
- There are several **identities**

Formula:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$

- **Quotient** relation (remember to **reciprocal** and **multiply** when **dividing** in **dividing**)

Example:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$$
$$\frac{\sin\theta}{\cos\theta} = \frac{y}{r}\left(\frac{r}{x}\right)$$
$$\frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$
$$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$$

- Watch for **square ratios**
- **Pythagorean identity**

Example:

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$
$$\sin^2\theta + \cos^2\theta = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2}$$
$$\therefore \sin^2\theta + \cos^2\theta = 1$$
$$\therefore \sin^2\theta = 1 - \cos^2\theta$$
$$\therefore \cos^2\theta = 1 - \sin^2\theta$$

- In order to solve certain **identities**, you'll need to find **common denominators** when given and **integer** of 1

- These 2 **identities** can prove other **identities**

Example:
$$\frac{\sin\theta\cos\theta}{\tan\theta} = 1 - \sin^2\theta$$
$$\frac{\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}} = 1 - \sin^2\theta$$
$$\sin\theta\cos\theta\left(\frac{\cos\theta}{\sin\theta}\right) = 1 - \sin^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$1 - \sin^2\theta = 1 - \sin^2\theta$$
$\because LS = RS$
$\therefore$ this is an identity

Example:
$$\frac{\tan^2\theta + 1}{\tan^2\theta - 1} = \frac{1}{\sin^2\theta - \cos^2\theta}$$
$$\frac{\left(\left(\frac{\sin^2\theta}{\cos^2\theta}\right) + 1\right)}{\left(\left(\frac{\sin^2\theta}{\cos^2\theta}\right) - 1\right)} = \frac{1}{\sin^2\theta - \cos^2\theta}$$
$$\frac{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}}{\frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}} = \frac{1}{\sin^2\theta - \cos^2\theta}$$
$$\frac{1}{\cos^2\theta}\left(\frac{(\cos^2\theta)}{\sin^2\theta - \cos^2\theta}\right) = \frac{1}{\sin^2\theta - \cos^2\theta}$$
$$\frac{1}{\sin^2\theta - \cos^2\theta} = \frac{1}{\sin^2\theta - \cos^2\theta}$$
$\because LS = RS$
$\therefore$ this is an identity

Example:
$$\sin x + \tan x = \tan x\,(1 + \cos x)$$
$$\sin x + \frac{\sin x}{\cos x} = \tan x\,(1 + \cos x)$$
$$\frac{(\sin x\cos x + \sin x)}{\cos x} = \tan x\,(1 + \cos x)$$
$$\frac{\sin x\,(\cos x + 1)}{\cos x} = \tan x\,(1 + \cos x)$$
$$\frac{\sin x}{\cos x}(\cos x + 1) = \tan x\,(1 + \cos x)$$
$\tan x\,(1 + \cos x) = \tan x\,(1 + \cos x) \because LS = RS$
$\therefore$ this is an identity

- Some trigonometric **identities** are a result of a definition, while others are derived from relationships
- **Reciprocal identities** are **identities** based on definitions
- **Cosecant (csc)**, **secant (sec)**, and **cotangent (cot)**, are identity names of certain **ratios**

Formula:
$$\csc\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$$
$$\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0 \cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$$

- **Quotient identities** are derived from relationships

Formula:
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$$

- **Pythagorean identities** are derived from relationships

Formula:
$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

- To prove that a given trigonometric equation is an **identity**, both sides of the equation need to be equal. There are several methods of doing so
- Simplifying the complicated side or manipulating both sides to get the same expression
- Rewriting all trigonometric **ratios** in term of x, y, and r
- Rewriting all expressions involving **tangent** and the **reciprocal** trigonometric **ratios** in terms of **sine** and **cosine**
- Applying the **Pythagorean identity** where appropriate
- Using a common **denominator** or **factoring** as required

Example:
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$
$$\frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$
$$1\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\cos\theta}{\sin\theta}$$
$$\frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}$$

- Keep on watch for alternative **ratios** that might be represented differently

Example:

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2$$
$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\frac{1}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Example:

$$1 + \cos^2\theta = \csc^2\theta$$
$$1 + \frac{1}{\tan^2\theta} = \left(\frac{1}{\sin\theta}\right)^2$$
$$1 + \frac{\frac{1}{\sin^2\theta}}{\cos^2\theta} = \frac{1}{\sin^2\theta}$$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$\frac{1}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Example:

$$\tan\theta + \cot\theta = \frac{1}{\sin\theta cos\theta}$$
$$\frac{\sin\theta}{\cos\theta} + \frac{cos\theta}{\sin\theta} = \frac{1}{\sin\theta\, cos\theta}$$
$$\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\, cos\theta}$$
$$\frac{1}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\, cos\theta}$$

- Recall **conjugates**, It may be necessary to solve in certain cases

Example: $\frac{\sin x}{1+\cos x} = \csc x - \cot x$

$$\frac{\sin x}{1+\cos x}\left(\frac{1-\cos x}{1-\cos x}\right) = \frac{1}{\sin x} - \frac{1}{\tan x}$$

$$\frac{\sin x\,(1-\cos x)}{(1+\cos x)(1-\cos x)} = \frac{1}{\sin x} - \frac{1}{\frac{\sin x}{\cos x}}$$

$$\left(\frac{\sin x\,(1-\cos x)}{1-\cos^2 x}\right) = \frac{1}{\sin x} - \frac{\cos x}{\sin x}$$

$$\frac{\sin x\,(1-\cos x)}{\sin^2 x} = \frac{1-\cos x}{\sin x}$$

$$\frac{1-\cos x}{\sin x} = \frac{1-\cos x}{\sin x}$$

Advanced Functions

Interval Notation

A **relation** is a set of **ordered** pairs. The **domain** of a **relation** is the set of first **elements** in the **ordered pair**. The **range** is a set of second **elements** in the **ordered pair**. A **function** is a **relation** in which each **element** of the **domain** is paired with one and only one **element** of the **range** (vertical line test).

- Three ways to represent a **function**
- **Numerically**; ordered pairs arranged in an x, y table
- **Algebraically**; expressed as $f(x)$ followed by a **domain** and **range**
- **Graphically**; on a Cartesian graph with plotted points
- **Power functions** in general are written in the form, $y = x^n$ where n is a whole number/**integer**
- **Polynomial functions** written with **constants** and **degrees** that must be whole numbers

Formula: $y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ \dots + a_1 + \ a_0$
a_0 is the **constant term**
n is the **degree** of the **polynomial** (whole number)

Example: $y = 6x^5 + 7x^4 - 5x^3 + 3$
∴ Degree = 5

- A **polynomial functions** must have whole numbers as **degree**

Example: $y = 6x^{-5} + 7x^4 - 5x^{-3} + 3$

Example: $y = 6x^0$

- New notation recognized as **interval notation**. Several cases are shown demonstrating the use of square and rounded brackets. Square brackets indicated an equal to and/or greater than/less than. Infinity symbol is used to signify the **function** continues and are always surrounded by rounded brackets

Example:

Old Notation	Interval notation
$\{x \in \mathbb{R} \mid -3 < x < 10\}$	$x \in (-3,10)$
$\{x \in \mathbb{R} \mid -3 \leq x \leq 10\}$	$x \in [-3,10]$
$\{x \in \mathbb{R} \mid x > 4\}$	$x \in (4, \infty)$
$\{x \in \mathbb{R} \mid x \geq 4\}$	$x \in [4, \infty)$
$\{x \in \mathbb{R} \mid x \leq 6\}$	$x \in (-\infty, 6]$
$\{x \in \mathbb{R}\}$	$x \in (-\infty, \infty)$

Power Functions

Functions that have an identifiable whole number as a **degree**.

- **Power functions** have given names associated with their **degree**

Power Function	Degree	Name
$y = a$	0	Constant
$y = ax$	1	Linear
$y = ax^2$	2	Quadratic
$y = ax^3$	3	Cubic
$y = ax^4$	4	Quartic
$y = ax^5$	5	Quintic
$y = ax^6$	6	Degree 6

- **Power functions** can relate between odd and even **degrees**
- **End behaviour** is if the **function's** extremities/ends and their location (**quadrant** wise). It is in the notation of if $y = \pm\infty$ and $x = \pm\infty$

Example: $y = x^3$
Left end is down $(x, y = -\infty)$
Right end is up $(x, y = +\infty)$
Extends from quadrant 3 to quadrant 1

Example: $y = x^4$
Left end is up $(x = +\infty; y = -\infty)$
Right end is up $(x, y = +\infty)$
Extends from quadrant 2 to quadrant 1

Example: $y = -3x^2$
Extends from quadrant 3 to quadrant 4
$\because$ of an even exponent, and negative coefficient

Example: $y = -\frac{2}{5}x^9$
Extends from quadrant 2 to quadrant 4
$\because$ of an odd exponent, and negative coefficient

Example: $y = 2x$
Extends from quadrant 3 to quadrant 1
$\because$ of an odd exponent, and positive coefficient

- Proper notation for **end behaviour** is comparing both the x and y endpoints and their quadrants. Expressed as x approcahes; notated by an arrow →, infinity

  Example: $y = x^3$
  as $x \to \infty, y \to \infty$ (As x approaches infinity, y approaches infinity)
  as $x \to -\infty, y \to -\infty$ (As x approaches negative infinity, y approaches negative infinity)

  Example: $y = -x^3 - x^2 + 4x + 4$
  as $x \to \infty, y \to -\infty$
  as $x \to -\infty, y \to \infty$

- A **graph** has **line symmetry** if the **graph** has a visible x-axis that divides the **graph** into 2 mirror parts

  Example: If $a = x$-axis
  $y = x^2 x = a$
  ∴ function has line symmetry

- A **graph** has **point symmetry** if the **graph** has **points** (a, b) **rotated** $180°$ and remains **congruent**

  Example: $y = x^3$
  ∴ function has point symmetry

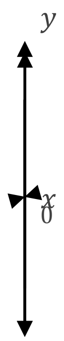
- Even and odd **power functions** share identical **end behaviour**, **symmetry** methods, **domain** and **range**

Feature	$y = x^n$, odd	$y = x^n$, even
Domain	$x \in (-\infty, \infty)$	$x \in (-\infty, \infty)$
Range	$y \in (-\infty, \infty)$	$y \in [0, \infty)$
Symmetry	Point symmetry	Line symmetry
End Behaviour	$x: -\infty \downarrow, +\infty \uparrow$	$x: -\infty \uparrow, +\infty \uparrow$

- **Graphs** can have a **minimum** number of points and a **maximum** number of points
- **Power functions** can has a **local minimum** and a **local maximum** points that are visible before they extend to infinity
- **Power functions** can have multiple x-**intercepts** depending on the **function** itself

- **Roots** of a **function** will also determine the **degree**
- A **root** is defined by how many times the **function** crosses the x-axis or intercepts
- There are 3 kinds of **roots**
- First **root** being a **real distinct root**, whereby the function crosses and clears the x-axis at one point
- Second **root** being a **real equal root**, whereby a **parabola** meets the x-axis
- Third **root** being an **imaginary root**, or **complex root**, whereby a **parabola** does not meet the x-axis
- If the **polynomial** has a Quintic (5) **degree**, there will be 5 **roots**

- From a **power function**, you can determine its alternate **graphical** or **algebraic** form recognizing if it has a positive or negative **coefficient**, its **end behaviours**, its **local minimum** and **maximum** points, it's x, y-intercepts, and its **symmetry** method
- **Graphically**, determine the total number of **local minimum** and **maximum** points. Once totalled, it can be determined that the leading **degree** is 1 higher than the total
- **Absolute maximum** and **minimum** refer to the **functions** infinite **end behaviour**, not local
- **Graphically**, depending on the location of its **end behaviour**, it can be determined whether or not the **leading coefficient** is positive or negative, and if the **degree** is odd or even



Example:

Has 2 local minimums and 2 local maximums

$\because$ total local points $= 4$, $\therefore$ degree $= 4 + 1 = 5$ or Quintic, thus an odd degree

Positive coefficients with odd degrees extend from quadrant 3 to quadrant 1

Power Functions Summary

Function	Linear	Quadratic	Cubic	Quartic	Quintic
Domain	$x = \epsilon(-\infty, \infty)$	$x = \epsilon(-\infty, \infty)$	$x = \epsilon(-\infty, \infty)$	$x = \epsilon(-\infty, \infty)$	$x = \epsilon(-\infty, \infty)$
Range	$y = \epsilon(-\infty, \infty)$	Varies	$x = \epsilon(-\infty, \infty)$	Varies	$x = \epsilon(-\infty, \infty)$
Max # of x-intercepts	1	2	3	4	5
Max # local min/max	0	1	2	3	4

Finite Differences

Finite differences for a **polynomial function** of **degree** n (positive **integer**), the nth differences are equal (or **constant**), have the same sign as the leading **coefficient**, and are equal to n **factorial**. Used typically **algebraically** or numerically.

- For a positive integer n, the product $n \times (n-1) \times \ldots \times 2 \times 1$ can be expressed as $n!$ or factorial

  Formula: $n!$

  Example: $5! = 5 \times 4 \times 3 \times 2 \times 1$
  $= 120$

- Given an **algebraic power function**, it can be determined which **finite difference** will be **constant** by the **function's degree**

  Example: $g(x) = -4x^3 + 2x - x + 5$
  $\therefore$ the 3rd finite difference will be constant

- Given an **algebraic power function**, it can be determined the value of the **constant finite difference** by $n!$ where n is a positive **degree** multiplied with the **leading coefficient**
- When referred to the **constant**, it is referring to the value of the **constant finite differences**
- Where c is the **constant**, a is the leading **coefficient**, and n is the **degree** of the **polynomial**

  Formula: $c = a(n!)$

  Example: $g(x) = -4x^3 + 2x - x + 5$
  $c = -4(3!)$
  $c = -4(6)$
  $c = -24$

- With **finite differences**, the value of the **constant** also has the same sign($\pm$) as the leading **coefficient** of the **polynomial**

Example: Given a fifth difference of 60, determine the degree and value of the leading coefficient

$\because$ the 5th difference is constant, the degree of the polynomial is Quintic (5)

$c = a(n!)$

$60 = a(5!)$

$\frac{60}{5!} = \frac{a(5!)}{5!}$

$\therefore a = \frac{1}{2}$

- **First differences** given a **table of values** or **numerically**, works on finding the difference that remains **constant** throughout the **table of values**. The **finite difference** that is **constant** will determine the **degree**, the value of that **finite difference** will determine the sign value of the **leading coefficient**

Example:

x	y	**1**	**2**	**3**	**4**
-2	-40				
-1	12	$12-(-40)=52$			
0	20	$20-12=8$	$8-52=-44$		
1	26	$26-20=6$	$6-8=-2$	$-2-(-44)=42$	
2	48	$48-26=22$	$22-6=16$	$16-(-2)=18$	$14-42=-24$
3	80	$48-80=32$	$32-22=10$	$10-16=-6$	$-6-18=-24$

$\because$ the 4th difference is constant, the degree of the polynomial is Quartic (4)

$\because$ the constant is negative, (-24), the leading coefficient will be negative

$\therefore a = \frac{-24}{4!} = -1$

Equations and Graphs of Polynomial Functions

By reading a graph, the least possible **degree** and sign of the **function** can be determined.

- **Functions** can come in different forms and not in typical form. This new form identifies the x-intercepts by solving each **bracketed term**. The **degree** can be determined by using **like terms** with the x or **graphically**. Also, the y intercept can be found by zeroing the x values. Leading **coefficient** can be determined by the **product** of the x-**coefficients**. **End behaviour** is determined by the **degree** and sign of the **leading coefficient**.

  Example: $y = x(x-3)(x+2)(x+1)$
  When $y = 0, x = -2, -1, 0, 3$
  Degree is Quartic (4) because the product of the x's is 4
  y-intercept $= (0-3)(0+2)(0+1) = -6$
  $a = 1 \times 1 \times 1 \times 1 = 1$
  ∵ degree is Quartic and the function has a positive leading coefficient, the function extends from quadrant 2 to 1

  Example: $y = -(2x+1)^3(x-3)$
  When $y = 0, x = -\frac{1}{2}\,(\text{order } 3), 3$
  Degree is Quartic (4) because the product of the x's is 4
  y-intercept $= -(2(0)+1)^3(0-3) = 3$
  $a = -1 \times 2^3 \times 1 = -8$
  ∵ degree is Quartic and the function has a negative leading coefficient, the function extends from quadrant 3 to 4

- **Intervals** can be segmented in a **power function**. The x-intercepts divide the x-axis into multiple intervals. If $y > 0$ then it is positive, otherwise if $y < 0$ it is negative. Can be done both **algebraically** and **graphically**

  Example: $y = x(x-3)(x+2)(x+1)$

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f(x)$	$+$	$-$	$+$	$-$	$+$

- An **order** of an x-intercept or **root** is determined by the x **factor** with respect to the **degree**

Example: Determine an equation the polynomial function given a Quartic (4), zeroes at -10 order $2, 10$ order 2, and passes through the point $(0, 26)$

x Factors: $(x+10)$ order $2, (x-10)$ order 2

$\because y = k(x-a)(x-b)(x-c) \ldots; y = k(x+10)^2(x-10)^2$

Substitute $(0, 26)$

$26 = k(10)^2(-10)^2$

$\frac{26}{10000} = \frac{10000k}{10000}$

$k = 0.0026$

$\therefore y = 0.0026(x+10)^2(x-10)^2$

- **Graphically**, an order will appear as a stand alone x^n

Example: Given the equation, $y = -(x+4)^2(x-1)(x-3)$

Degree = Quartic (4)

$\because y = -1(4)^2(-1)(3)$

$= -48$

Odd and Even Functions

Graphically and **algebraically** indentify **symmetry**

- Recall that some odd **degree power functions** have a **point of symmetry**, and some **even degree power functions** have a **line of symmetry**
- **Polynomial functions** can be classified as an **even** or **odd function**
- All **even functions** have a **line of symmetry** about the y-axis $(x = 0)$
- All **odd functions** have a **point of symmetry** about the origin $(0,0)$
- An **even function** is a mirror image of itself with respect to the y-axis. If $f(x)$ is an **even function**, then $f(-x) = f(x)$

Example: $f(x) = 2x^2 - 3$
Test: $f(-x)$
$f(-x) = 2(-x)^2 - 3$
$f(-x) = 2x^2 - 3$
$\because f(x) = f(-x)$
$\therefore f(x)$ is an even function

- An **odd function** is rotationally symmetric about the origin. If the **graph** is rotated $180°$ about the origin, it does not change. If $f(x)$ is an **odd function**, then $f(-x) = -f(x)$

Example: $h(x) = -2x^3 + x$
Test: $h(-x)$
$h(-x) = -2(-x)^3 + (-x)$
$h(-x) = 2x^3 - x$
$\because h(x) \neq h(-x)$
$\therefore h(x)$ is not an even function
$\because -h(x) = h(-x)$
$\therefore h(x)$ is an odd function

- Neither is also a possibility

Example: $g(x) = -4x^2 + 3x - 2$
Test: $g(-x)$
$g(-x) = -4(-x)^2 + 3(-x) - 2$
$g(-x) = -4x^2 - 3x - 2$
$\because g(x) \neq g(-x)$, and $-g(x) \neq g(-x)$
$\therefore$ the function is neither an even function nor odd

Transformations of Power Functions

Recall from previous **equations**, except now **functions** will have **degrees**

- Double bars surrounding a **term** or **variable** indicates an **absolute** value, or the positive value only

  Example: $g(x) = -2f[3(x-2)] + 1$
  $|a| = 2$

- Given $y = f(x)$, then $g(x) = af[k(x-d)] + c$ is a **transformed function** of $f(x)$

  Formula: $g(x) = af[k(x-d)] + c$

  $a < 0 =$ reflection about the x-axis
  $0 < |a| < 1 =$ vertical compression by a factor of a
  $|a| > 1 =$ vertical stretch by a factor of a

  $k < 0 =$ reflection about the y-axis
  $0 < |k| < 1 =$ horizontal expansion by a factor of $\frac{1}{k}$
  $|k| > 1 =$ horizontal compression by a factor of $\frac{1}{k}$

  $d > 0 =$ horizontal shift right (fully factored)
  $d < 0 =$ horizontal shift to the left (fully factored)

  $c > 0 =$ vertical shift up
  $c < 0 =$ vertical translation down

- d may already be **factored** therefore in order to find the true **horizontal shift** you must **factor** the **term** with k (watch the brackets)

  Example: $g(x) = -2f\left[\frac{1}{3}x + 1\right] - 4$
  $g(x) = -2f\left[\frac{1}{3}(x+3)\right] - 4$
  $\therefore d = 3$, Shift left by 3

- Given a **function**, **transform** the **function** in the order of **stretches**, **reflections**, and **translations** or SRT
- When $g(x)$ is expected to be rewritten as a full **transformation**, rewrite it in **standard form**; or **expand** the **function**
- In order to work **graphically**, get a **table of values** set up and apply the **transformations** to a select **relation**

Example: Given $f(x) = x^4$

$$g(x) = -2f\left[\frac{1}{3}x + 1\right] - 4$$

$$g(x) = -2f\left[\frac{1}{3}(x + 3)\right] - 4$$

Describe the transformation: Reflection on x-axis, vertical stretch by a factor of 2, horizontal expansion by a factor of 3, shift left by 3, vertical translation down by 4

Rewrite $g(x)$ (Not standard form)

$$g(x) = -2\left(\frac{1}{3}x + 1\right)^4 - 4$$

Sketch $g(x)$

$f(x)^4$		$g(x) = -2f\left[\frac{1}{3}x + 1\right] - 4$	
x	y	x	y
x	$y = x^4$	$3x - 3$	$-2y - 4$
−3	81	−12	−166
−2	16	−9	−36
−1	1	−6	−6
0	0	−3	−4
1	1	0	−6
2	16	3	−36
3	81	6	−166

- Place $g(x)$ into $f(x)$ proportionally

Example: Given $f(x) = -3(x-1)^4 + 2$

$$g(x) = 2f\big(2(x+1)\big) - 3$$
$$g(x) = 2f(2x+2) - 3$$
$$f(x) = 2(-3\left(\frac{1}{2}(2x+2-1)^4 + 2\right) - 3$$
$$f(x) = 2\left(-3\left(x+\frac{1}{2}\right)^4 + 2\right) - 3$$
$$f(x) = -6\left(x+\frac{1}{2}\right)^4 + 1$$

Polynomial Division

Long division will be needed for this and recognizing the structure of how long division works is key

- **Polynomial** multiplication is when you **expand**

Example: $(x-3)(x^2-2x+5)$
$x^3-2x^2+5x-3x^2+6x-15$
$x^3-5x^2+11x-15$

- **Polynomial** division can be done by **factoring**

Example: $\frac{x^2-x-12}{x-4}$
$\frac{(x-4)(x+3)}{x-4}$
$x+3, x \neq 4$

- Long division has a **dividend** which is the first **term**, and a **divisor**, which is the second **term**. The **quotient** is the result of a division expression. The **divisor** is put up against each digit to see how many times the **divisor** divides fully into the first digit. The result is placed on top, and value is placed below the corresponding digit. The difference of the value and the first digit are placed below and the second digit is brought down. The processed is repeated.

Example: $876 \div 7; 876\text{:Dividend}, 7\text{:Divisor}$

$$\begin{array}{r|ccc} & 1 & 2 & 5 \\ \hline 7 & 8 & 7 & 6 \\ & \underline{7} & & \\ & \underline{1} & 7 & \\ & 1 & \underline{4} & \\ & & \underline{3} & 6 \\ & & 3 & \underline{5} \\ & & & 1 \end{array}$$

$\therefore \text{quotient} = 125, \text{Remainder } 1$
$876 = (7 \times 125) + 1$

- When working with **polynomial** division and you can't **factor**, focus only on x and its **degree**, see what you can do to x to match the **degree** of the **dividend**. When the remainder has a **degree** less than the **divisor**, it becomes the actual remainder

Example: $x^2-7x-10 \div x+2$
$\therefore x^2-7x-10 = (x+2)(x-9)+8$

Example: $3x^4-2x^3-7x+4 \div x^2-3x+1$
$\therefore 3x^4-2x^3-7x+4 = (x^2-3x+1)(3x^3+7x+18)+(40x-14)$

- A result in **quotient** form is when the **dividend** is physically expressed as a result of **dividend** over **divisor**. Then equalling the result form. A corresponding statement to check the division is writing the **quotient** out fully (product of divisor and **quotient** summed with the remainder). Verifying the answer means you expand the **quotient**

Example: $x^3 + 3x^2 - 2x + 5 \div x + 1$

$$\therefore \frac{x^3 + 3x^2 - 2x + 5}{x + 1} = x^2 + 2x - 4 + \left(\frac{9}{x+1}\right) \text{ Quotient Form}$$

$$x \neq -1 (x^2 + 2x - 4)(x - 1) + 9 \text{ Corresponding Statement}$$

$$= x^3 + x^2 + 2x^2 + 2x - 4x - 4 + 9$$

$$= x^3 + 3x^2 - 2x + 5$$

Example: $3x^4 - 4x^3 - 6x^2 + 17x - 8 \div 3x - 4$

$$\therefore \frac{3x^4 - 4x^3 - 6x^2 + 17x - 8}{3x - 4} = x^3 - 2x + 3\left(\frac{4}{3x-4}\right) \text{ Quotient Form}$$

$$x \neq \frac{4}{3}(x^3 - 2x + 3)(3x - 4) + 4 \text{ Corresponding Statement}$$

$$= 3x^4 - 4x^3 - 6x^2 + 8x + 6x - 12 + 4$$

$$= 3x^4 - 4x^3 - 6x^2 + 17x - 8$$

- **Synthetic division** can only be used if the **divisor** is in the form $(x + c), c \in \mathbb{R}$. Place the x **factor** aside from the **constants** of $f(x)$. Place a 0 in missing **degrees**. Multiply the first **constant** and the **factor** and add the resultant to the second **constant** and so on. The last sum is the **remainder** and the resultants are the new **coefficients** starting 1 less **degree** than the **dividend**

The Remainder Theorem

When a **polynomial** $P(x)$ is divided by $(x - b)$, then the remainder is $P(b)$ or when $P(x)$ is divided by $(ax - b)$, then the remainder is $P\left(\frac{b}{a}\right)$

- **Polynomial** division can be written in a form whereby the remainder is given and other **constants** can be solved

  Formula: $P(x) = D(x)Q(x) + R(x)$
  $D = \text{divisor}$
  $Q = \text{Quotient}$
  $R = \text{Remainder}$
  or
  $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$

- When given the **divisor**, b is the **factor** of x Therefore $P(b) = r$

  Example: $P(x): 2x^3 - 2x^2 - 3x + 3 \div x - 3$
  $b = 3$
  $2x^2 + 4x + 9 + \frac{30}{x - 3}$ Quotient Form
  $P(3) = 30$

- Formula proof (watch x) expressed **algebraically**

  Formula: $P(x) = D(x)Q(x) + R(x)$
  $P(x) = (x - b)Q(x) + R(x)$
  $P(b) = (b - b)Q(x) + R(x)$
  $P(b) = R(x)$

- The remainder can be determined by subbing in the **factor** of the divisor into x

  Formula: $x^3 + 3x^2 - 2x - 1 \div x + 1$
  $\therefore b = -1; P(-1)$
  $(-1)^3 + 3(-1)^2 - 2(-1) - 1 = 3$

- Determine k

Example: $P(x) = x^3 - 4x^2 + kx - 1 \div (2x - 3); r = \frac{7}{8}$, determine

$\therefore b = \frac{3}{2}; P\left(\frac{3}{2}\right)$

$P\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 4\left(\frac{3}{2}\right)^2 + k\left(\frac{3}{2}\right) - 1 = \frac{7}{8}$

$P\left(\frac{3}{2}\right) = \frac{7}{8}$

$\frac{7}{8} = \frac{27}{8} - 4\left(\frac{9}{4}\right) + \frac{3k}{2} - 1$

$\frac{7}{8} - \frac{27}{8} = -9 + \frac{3k}{2} - 1$

$-\frac{5}{2} + 10 = \frac{3k}{2}$

$-\frac{10}{4} + \frac{40}{4} = \frac{3k}{2}$

$\frac{40}{4} = \frac{3k}{2}$

$2\left(\frac{40}{4}\right) = 2\left(\frac{3k}{2}\right)$

$\frac{60}{4} = 3k$

$15 = 3k$

$k = 5$

- Substitution or elimination may be required given more of some information and less of another

Example: $P(x) = x^3 + 3x^2 - mx + n \div x - 5; r = 15$ when divided by $x - 2, r = -48$. Determine m, n

$P(5) = 15$
$15 = (5)^3 + 3(5)^2 - m(5) + n$
$15 = 125 + 75 - m(5) + n$
$-185 = -m(5) + n$

$P(2) = -48$
$-48 = (2)^3 + 3(2)^2 - m(5) + n$
$-48 = 8 + 12 - 2m + n$
$-68 = -2m + n$

$-185 = -5m + n$
$-(-68 = -2m + n)$ Elimination
$-117 = -3m$
$\therefore m = 39$

$-185 = -5(39) + n$
$-185 = -195 + n$
$\therefore n = 10$

The Factor Theorem

States a **polynomial** $P(x)$ has a factor $(x - b)$, if and only if (iff) $P(b) = 0$. Therefore if $r = 0$, the **divisor** is a **factor** of the **dividend**. Similarly, a polynomial $P(x)$ has a factor $(ax - b)$, iff $P\left(\frac{b}{a}\right) = 0$

- Only using terms

  Example: $24 \div 6 = 4$
  $\therefore$ 6 is a factor of 24

- Testing a given **term** can determine if the **divisor** is a **factor** of the **dividend** if $r = 0$

  Example: Is $(x + 2)$ a factor of $f(x) = x^3 + 3x^2 + 5x + 9$
  $b = -2$
  $f(b) = (-2)^3 + 3(-2)^2 + 5(-2) + 9$
  $r = -8 + 12 - 10 + 9$
  $r = 3$
  $\because f(-2) \neq 0$, by factor theorem, $(x + 2)$ is not a factor of $f(x)$

- In order to properly **factor** a **polynomial**, using the **remainder** and **factor** theorems you can simplify. Testing a possible **factor** may result in a remainder of 0

  Formula: Factor $f(x) = x^3 - 7x + 6$
  Let $b = -1 \rightarrow f(-1) = 12 \therefore b \neq -1$
  Let $b = 1 \rightarrow f(1) = 0 \therefore (x - 1)$ is a possible factor
  $x^3 - 7x + 6 \div x - 1$
  $= x^2 + x - 6$
  $\therefore (x - 1)(x^2 + x - 6)$
  $= (x - 1)(x + 3)(x - 2)$

Integral and Rational Zero Theorem

If $x = b$ is an **integral zero** of the **polynomial** with **integral coefficients**, then b is a factor of the **constant term** of the **polynomial**

- Recognize the **constant** of the **polynomial** and find **factors** that could result in it (positive or negative). These are all possible **factors** or test values to result in $r = 0$

Example:

$f(x) = x^3 - x^2 - 14x + 24$
Constant $= 24$
$\therefore b \pm = 1, 2, 3, 4, 6, 8 \ldots$

- Test the possible **factors**

Example:

$f(x) = x^3 - x^2 - 14x + 24$
Test: Let $b = 2$
$f(2) = (2)^3 - (2)^2 - 14(2) + 24$
$f(2) = 0 \therefore (x - 2)$ is a factor
$x^3 - x^2 - 14x + 24 \div x - 2$
$= x^2 + x - 12$
$\therefore f(x) = (x - 2)(x^2 + x - 12)$
$= (x - 2)(x + 4)(x - 3)$

- If $x = \frac{b}{a}$ is a **rational zero** of the **polynomial** $P(x)$ with **integral coefficients**, then b is a **factor** of the **constant term** of the **polynomial** and a is a **factor** of the leading **coefficient**

Example:

Factor: $P(x) = 6x^3 - x^2 - 9x - 10$
$b \pm = 1, 2, 5, 10$
$a \pm = 1, 2, 3, 6$
Test: $\frac{b}{a} \pm = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 2, \frac{2}{3} \ldots$
$P\left(\frac{5}{3}\right) = 0 \therefore (3x - 5)$ is a factor
$2x^2 + 3x + 2 \div 3x - 5$
$= 2x^2 + 3x + 2$
$\therefore f(x) = (3x - 5)(2x^2 + 3x + 2)$
Cannot further factor

Families of Polynomial Functions

A family of **functions** is a set of **functions** that have the same zeros or x-intercepts but have different y-intercepts (unless zero is one of the x-intercepts)

- An equation for the family of polynomial functions with zeros a_n

  Formula: $y = k(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$
  $k \in \mathbb{R}, k \neq 0$

- Given the **degree** and zeroes of a family, equations for the **function** can be determined by getting the **factor** of the zeroes and alternative members of the families can be discovered by subbing in a value for k.
- In order to find a member whose **graph** passes through a given point, substitute the values into the original **equation** and solve for k
- Represent a family of functions **algebraically**

  Example: Zeroes of a family of a quadratic function are 2 and -3
  $\because$ factors are $(x - 2)$ and $(x + 3)$
  $\therefore y = k(x - 2)(x + 3)$
  Let $k = 8; y = 8(x - 2)(x + 3)$; an family member
  Find a member whose points pass through $(1,4)$
  $4 = k(1 - 2)(1 + 3)$
  $4 = k(-1)(4)$
  $4 = -4k$
  $k = -1$
  $\therefore y = -(x - 2)(x + 3)$

- The y-intercept can be treated as another point, therefore substitute x as 0 and the y value correspondingly

  Example: Zeroes of a family of a cubic function are $-2, 1$ and 3
  $\therefore y = k(x + 2)(x - 1)(x - 3)$
  Find a member whose points y-intercept $= -15$
  $-15 = k(0 + 2)(0 - 1)(0 - 3)$
  $k = -2.5$
  $\therefore y = -2.5(x + 2)(x - 1)(x - 3)$

- When working with irrational zeroes, recall difference of squares: $(a-b)(a+b)=a^2-b^2$

Example: Zeroes of a family of a quartic function are $1, -1, 2+\sqrt{3}$ and $2-\sqrt{3}$

$$\therefore y = k(x-1)(x+1)\left(x-2-\sqrt{3}\right)\left(x-2+\sqrt{3}\right)$$
$$= k(x-1)(x+1)\left[(x-2)-\sqrt{3}\right]\left[(x-2)+\sqrt{3}\right]$$
$$= k(x^2-1)\left[(x-2)^2-\left(\sqrt{3}\right)^2\right]$$
$$= k(x^2-1)(x^2-4x+4-3)$$
$$= k(x^2-1)(x^2-4x+1)$$
$$= k(x^4-4x^3+x^2-x^2+4x-1)$$
$$= k(x^4-4x^3+4x-1)$$

Solving Polynomial Equations

Recall how the **degree** effects the number and kinds **roots** a **polynomial functions** have.

- Recall factoring and solving for x

Example: $5x + 4 = 0$
$5x = -4$
$x = -\frac{4}{5}$

Example: $x^2 - x - 12 = 0$
$(x-4)(x+3) = 0$
$\therefore x = 4, -3$

- **Quadratic equation** can be used to solve **polynomials** that can't be factored in order to find **roots** or **imaginary roots**
- Recall **common factoring**

Example: $x^3 - 4x^2 - 12x = 0$
$x(x^2 - 4x - 12) = 0$
$x(x-6)(x+2) = 0$
$\therefore x = 0, 6, -2$

- The **factor theorem** and **integral zero theorem** can be applied as well

Example: $x^3 - 3x^2 - 4x + 12 = 0$
Let $P(x) = x^3 - 3x^2 - 4x + 12$
Test: $P(b) \pm = 1, 2, 3, 4, 6, 12$
$P(2) = 0$
$\therefore$ $(x-2)$ is a factor of P(x); Then divide to fully factor
or
Factor by grouping
$(x^3 - 3x^2) - (4x - 12) = 0$
$x^2(x-3) - 4(x-3) = 0$
$(x^2 - 4)(x-3) = 0$
$(x-3)(x-2)(x+2) = 0$
$\therefore x = 3, \pm 2$

- **Difference of cubes**

Formula: $a^3 - b^3 = (a-b)(a^2 - ab + b^2)$

Example:
$x^3 - 27$
$(x)^3 - (3)^3$
$(x-3)(x^2 - 3x + 9)$

- **Sum of cubes**

Formula: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Example:
$64x^3 - 81$
$(4x)^3 + \left(\sqrt[3]{81}\right)^3$
$\left(4x + \sqrt[3]{81}\right)\left(16x^2 - 4\sqrt[3]{81}x + 81^{\frac{3}{2}}\right)$
Alternative forms: $81^{\frac{3}{2}} = \left(\sqrt[3]{81}\right)^2 = \left(81^{\frac{1}{3}}\right)^2$

- **Factor sum of cubes**

Example:
$x^3 + 1 = 0$
$(x+1)(x^2 - x + 1) = 0$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2a}$
$x = \frac{1 \pm \sqrt{-3}}{2}$
$x = \frac{1 \pm i\sqrt{3}}{2}$
$x = -1, \frac{1 \pm i\sqrt{3}}{2}$

- **Factor sum of cubes**

Example: $6x^3 - 13x^2 + x + 2 = 0$

Let $P(x) = 6x^3 - 13x^2 + x + 2$

$b\pm = 1, 2$

$a\pm = 1, 2, 3, 6$

Test: $P\left(\frac{b}{a}\right), P(2) = 0$

$\therefore P(x) \div (x - 2)$

Let $f(x) = 6x^2 - x - 1$

$6x^3 - 13x^2 + x + 2 = 0$

$(x - 2)(6x^2 - x - 1) = 0$

$(x - 2)(2x - 1)(3x + 1) = 0$

$\therefore x = 2, \frac{1}{2}, -\frac{1}{3}$

Polynomial Inequalities

Recognizing **polynomial intervals** and when $y <> 0$. Can be done both **graphically** and **algebraically**.

- A change in direction can be identified at local **minimum** and **maximum** points. Plugging in values will help you graph a **polynomial function**

  Example: $f(x) = (x+2)(x-2)(x-1)$
  $\therefore x = -2, 2, 1$
  $f(0) = (0+2)(0-2)(0-1)$
  $\therefore y = 4$
  $f(1.5) = -0.875$ Change in direction
  $f(-1) = 6$ Change in direction
  $f(-3) = -20$ Visible left most point
  $f(3) = 10$ Visible right most point
  $x \in (-\infty, -2), f(x) < 0$
  $x \in (-2,1), f(x) > 0 x \in (1,2), f(x) < 0 x \in (2, \infty), f(x) > 0$

- Solving a **linear inequality**. Treat the $<, >$ signs as $=$ signs and solve for x. A change in direction occurs when **multiplying** or **dividing** by a negative such of that in the last step of the example. The **inequality** can also be represented on a number line

  Example: $5 - 2x < 8$
  $-2x < 3$
  $x > -\frac{3}{2}$

- Solve a **quadratic inequality**. When multiplying 2 **factors** to get a positive result ($f(x) > 0$), their signs must be the same, therefore both positive or both negative. Determine the **intervals** where each **factor** is positive or negative. Find the **zeroes**, set-up **intervals**, and then test each value

  Example: $(x+2)(x-1) > 0$

Interval	$(-\infty, -2)$	$(-2,1)$	$(1, \infty)$
$(\boldsymbol{x}+\mathbf{2})$	$-$	$+$	$+$
$(\boldsymbol{x}-\mathbf{1})$	$-$	$-$	$+$
Sign of $\boldsymbol{f}(\boldsymbol{x})$	$+$	$-$	$+$

$\therefore x \in (-\infty, -2) \cup (1, \infty)$

- Solve a **polynomial inequality**

Example: $x^3 - 5x^2 + 2x + 8 \leq 0$

Interval	$(-\infty, -1)$	$(-1,2)$	$(2,4)$	$(4, \infty)$
$(\boldsymbol{x+1})$	$-$	$+$	$+$	$+$
$(\boldsymbol{x-2})$	$-$	$-$	$+$	$+$
$(\boldsymbol{x-4})$	$-$	$-$	$-$	$+$
Sign of $\boldsymbol{f}(\boldsymbol{x})$	$-$	$+$	$-$	$+$

$\therefore x \in (-\infty, -1] \cup [2,4]$

- Solve a **polynomial inequality** graphically

Example: $2x^3 - 3x^2 - 9x + 5 < 0$
$x \cong -1.79, 0.5, 2.79$
$\therefore x \in (-\infty, -1.79) \cup (0.5, 2.79)$

Example: $x^3 - 5x + 4 \geq 0$
$x \cong -2.56, 1.56$
$\therefore x \in [-2.56, \infty)$

Rational Functions

A **rational functions** has the form $h(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are **polynomials**

- **Domain** of a **rational function** are all **real numbers** except for when $g(x) = 0$
- The **zeroes** of $h(x)$ are equal to $f(x)$

Example: $h(x) = \frac{x}{x-4}$

$x \in (-\infty, 4) \cup (4, \infty)$

$x = 0$ (numerator)

- **Vertical asymptotes** (V.A.) can be found by setting x so that the **denominator** results in 0
- **Horizontal asymptotes** (H.A.) found by charting large values of x approaching y

Example: $f(x) = \frac{2}{x+2}$

Domain: $x \in (-\infty, -2) \cup (-2, \infty)$

Zeroes: $2 \neq 0; \therefore$ no zeroes

V.A: $x = -2$

H.A Set up a table where $x \to \pm\infty$, and see where y is approaching

H.A: $y \to 0.0007$

- Simplify when possible but refer to the original **function** for key features

Example: $f(x) = \frac{x-2}{x^2-2x} = \frac{x-2}{x(x-2)} = \frac{1}{x}$

$x \in (-\infty, 0) \cup (0, 2) \cup (2, \infty)$

$1 \neq 0; \therefore$ no zeroes

V.A: $x = 0$

H.A: $y \to 0$ (Test Values)

$\boldsymbol{x \to -\infty}$	y	$x \to \infty$	y
$\mathbf{-100}$	-0.01	100	0.01
$\mathbf{-1000}$	-0.001	1000	0.001
$\mathbf{-10000}$	-0.0001	10000	0.0001

- To express the **end behaviour** of a **rational function**, use the notation of x approaching the **vertical asymptote** from both the left $-\infty$, and the right ∞. Use small test values to determine positive or negative infinity values. To express left, a^- and right, a^+. This meaning as it gets closer to the **vertical asymptote**, what is happening to y

Example: $f(x) = -\frac{3}{x-1}$
V.A: $x = 1$

$x \to 1^-$	y	$x \to 1^+$	y
0.9	30	1.1	-30
0.99	300	1.01	-300
0.999	3000	1.001	-3000

as $x \to 1^-, y \to \infty$
as $x \to 1^+, y \to -\infty$

- The graph of a **rational function** has at least one asymptote, which maybe vertical, horizontal, or oblique
- An **oblique asymptote** is neither vertical or horizontal
- The graph of a **rational function** never crosses a **vertical asymptote** but it may/may not cross a **horizontal asymptote**
- The **reciprocal of a linear function** has the form $f(x) = \frac{1}{kx-c}$
- The restriction on the **domain** of a **reciprocal linear function** can be determined by finding the value of x that makes the denominator equal to zero, that is, $x = \frac{c}{k}$
- The **vertical asymptote** of a **reciprocal linear function** has an equation of the form $x = \frac{c}{k}$
- The **horizontal asymptote** of a **reciprocal linear function** has the equation $y = 0$
- If $k > 0$, the left branch of a **reciprocal linear function** has a negative, decreasing slope, and the right branch has a negative, increasing slope
- If $k > 0$, the left branch of a **reciprocal linear function** has a positive, increasing slope, and the right branch has a positive, decreasing slope

- **Reciprocal** of a **quadratic function** has a **degree** of 2
- Key features include: **domain, x-intercepts, y-intercept, vertical asymptotes, end behaviour, and horizontal asymptotes**
- Simplify where possible

Example: $f(x) = \frac{3}{x^2-4} = \frac{3}{(x-2)(x+2)}$

D: $x \in (-\infty, -2) \cup (-2,2) \cup (2, \infty)$

x-intercepts: $y = 0; 0 = \frac{3}{x^2-4} \because 3 \neq 0 \therefore$ none

y-intercepts: $x = 0; f(0) = \frac{3}{0^2-4}; y = -\frac{3}{4}$ or -0.75

V.A: $x = \pm 2$End Behaviour:

End behaviour for $x = 2$				End behaviour for $x = -2$			
$x \to 2^-$	y	$x \to 2^+$	y	$x \to -2^-$	y	$x \to -2^+$	y
1.9	-7.69	2.1	7.31	-2.1	7.31	-1.9	-7.69
1.99	-75.19	2.01	74.81	-2.01	74.8	-1.99	-75.19
1.999	-750	2.001	748	-2.001	749	-1.999	-750

as $x \to 2^-, y \to -\infty$
as $x \to 2^+, y \to \infty$
as $x \to -2^-, y \to \infty$
as $x \to -2^+, y \to -\infty$
H.A:(Numerically, sub in large values of x)

$x \to -\infty$	y	$x \to \infty$	y
-100	0	100	0
-1000	0	1000	0
-10000	0	10000	0

H.A: $y = 0$

- **Vertical asymptotes** are always dealt with the **denominator**
- **Horizontal asymptotes** are dealt by using large values of x to see the value y approaches
- The zeroes are determined by the **numerator**

Example: $f(x) = \frac{2x-7}{5x+3}$

$x \in \left(-\infty, -\frac{3}{5}\right) \cup \left(-\frac{3}{5}, \infty\right)$

$-\frac{7}{2} \neq 0; \therefore$ no zeroes

V.A: $x = -\frac{3}{5}$

H.A: $y \to 0.4$ (Test Values, Numerically)

$x \to -\infty$	y	$x \to \infty$	y
−100	0.416	100	0.383
−1000	0.401	1000	0.398
−10000	0.400	10000	0.0399

- The **horizontal asymptote** can be determined **algebraically** recognizing that given $y = \frac{1}{x}$, as $x \to \pm\infty, y \to 0$. Divide each term with x^n (highest **degree**)

Example: $f(x) = \frac{2x-7}{5x+3}$

$$y = \left(\frac{\frac{2x}{x} - \frac{7}{x}}{\frac{5x}{x} + \frac{3}{x}}\right) = \frac{2 - \frac{7}{x}}{5 + \frac{3}{x}}; \text{as } x \to \pm\infty, y \to \frac{2 - \frac{7}{x}}{5 + \frac{3}{x}} \approx \frac{2}{5} = 0.4$$

The result is equal to the previous example

- **Oblique asymptotes** or linear asymptotes occur in **rational functions** when the **degree** of the **numerator** is greater by 1 than the **degree** of the **denominator**
- To determine the equation of the **oblique asymptote**, use long division. The **quotient** will be the **oblique asymptote**

Example: $f(x) = \frac{(2x^3 - x^2 + 3)}{x^2}$

V.A: $x = 0$

$\therefore f(x) = (2x - 1) + \frac{3}{x^2}$ (result of long division, 3 is the remainder)

as $x \to \pm\infty, f(x) \to 2x - 1$

$\therefore$ O.A: $y = 2x - 1$

Solving Rational Equations

Solve for x-intercepts

- Solve through **algebraically** by **factoring** or **quadratic equation**

Example:

$$\frac{x}{2x-8}=3$$
$$\frac{x}{2x-8}=\frac{3}{1}$$
$$x=3(2x-8)$$
$$x=6x-24$$
$$5x=24$$
$$\therefore x=\frac{24}{5}$$

Example:

$$-\frac{4}{x-1}=\frac{7}{2-x}+\frac{3}{x+1}$$
$$-\frac{4}{x-1}=\frac{7(x+1)+3(2-x)}{(2-x)(x+1)}$$
$$-\frac{4}{x-1}=\frac{7x+7+6-3x}{2x+2-x^2-x}$$
$$-\frac{4}{x-1}=\frac{4x+13}{x^2+x+2}$$
$$-4(-x^2+x+2)=(x-1)(4x+13)$$
$$4x^2-4x-8=4x^2+13x-4x-13$$
$$-4x-8=9x-13$$
$$-13x=-5$$
$$\therefore x=\frac{5}{13}$$

Example:

$$\frac{1}{x^2-2x-7}=1$$
$$1=x^2-2x-7$$
$$x^2-2x-8=0$$
$$(x-4)(x+2)=0$$
$$\therefore x=4,-2$$

Solving Rational Inequalities

Similar to solving **polynomial inequalities**. **Numerator** of a **rational inequality** are the x-intercepts and **denominators** are the **restrictions** or **asymptotes**.

- Zeroes are found from the **numerators** and the undefined points are found from the **denominator**. From these values, you receive the **intervals**, then find the sign of $f(x)$ at each **interval**

Example: $\frac{x^2+3x+2}{x^2-16} \geq 0$

$\frac{(x+2)(x+1)}{(x+4)(x-4)} \geq 0$

Numerator (Zeroes): $x = -2, -1$

Denominator (Undefined): $x = -4, 4$

Intervals:$(-\infty, -4), (-4, -2), (-2, -1), (-1,4), (4, \infty)$

Interval	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1,4)$	$(4, \infty)$
$(x+1)$	$-$	$-$	$-$	$+$	$+$
$(x+2)$	$-$	$-$	$+$	$+$	$+$
$(x+4)$	$-$	$+$	$+$	$+$	$+$
$(x-4)$	$-$	$-$	$-$	$-$	$+$
Sign of $f(x)$	$+$	$-$	$+$	$-$	$+$

$x \in (-\infty, -4) \cup [-2, -1] \cup (4, \infty)$

Example: $\frac{2x^2+4x-30}{(x^2+5)(x^2-4x+4)} < 0$

$\frac{2(x+5)(x-3)}{(x^2+5)(x-2)^2} < 0$

Numerator (Zeroes): $x = -5,3$

Denominator (Undefined):no solution or $x = 2$

Interval	$(-\infty, -4)$	$(-4, -2)$	$(-2, -1)$	$(-1,4)$
2	$+$	$+$	$+$	$+$
$(x+5)$	$-$	$+$	$+$	$+$
$(x-3)$	$-$	$-$	$-$	$+$
(x^2+5)	$+$	$+$	$+$	$+$
$(x-2)^2$	$+$	$+$	$+$	$+$
Sign of $f(x)$	$+$	$-$	$-$	$+$

$x \in (-5,2) \cup (2,3)$

Special Case

Special case **rational functions** occurs when a numerator **factor** and a denominator **factor** eliminate each other. A hole in the graph appears at the x value of the eliminated **factor**, and the y value of the x substituted into the **function**. Recognized as a **discontinuity**.

- Factor the numerator and denominator appropriately, then eliminate

Example: $g(x) = \frac{2x^2-7x-4}{2x^2+5x+2}$

$$g(x) = \frac{(2x+1)(x-4)}{(2x+1)(x+2)};\ x \neq -2, -\frac{1}{2}$$

$$g(x) = \frac{x-4}{x+2}$$

There is a hole at the point $(-\frac{1}{2}, -3)$

Example: $f(x) = \frac{x^2-x-6}{x+2}$

$$f(x) = \frac{(x-3)(x+2)}{x+2}$$

$f(x) = x - 3; x \neq -2$There is a hole at the point $(-2, -5)$

Radian Measure

Radian measure is the standard for measuring angles. Alternative method to **degrees**.

- 1 **radian** is the measure of the angle subtended at the center of a circle by an arc equal in length to the radius of the circle
- Number of radians is the arc length divided by the radius

Formula: $\theta = \frac{a}{r}$

- Relationship between degrees and radian measure: $\theta = 360° \to \frac{\text{arc length}}{r}$, or the circumference of the whole circle. Therefore, $\theta = \frac{2\pi r}{r} = 2\pi(\text{rad}) = 360°$

Example: $1° = \frac{\pi}{180°}(\text{rad})$

Example: $1\ (\text{rad}) = \frac{180°}{\pi}$

Example: $45° \to (\text{rad})$

$$\frac{\pi}{180}\left(\frac{45}{1}\right) = \frac{45\pi}{180} = \frac{\pi}{4}$$

Example: $200° \to (\text{rad})$

$$\frac{\pi}{180}(200) = \frac{10\pi}{9}$$

Example: $\frac{2\pi}{3} \to (\text{degrees})$

$$\frac{180}{\pi}\left(\frac{2\pi}{3}\right) = 120°$$

Example: $2.3\ (\text{rad}) \to (\text{degrees})$

$$\frac{180}{\pi}(2.3) = 131.8°$$

- Similar to **degrees**, **radians** also has special angles and triangles
- List of special angles

Degrees	Radians
$\mathbf{30°}$	$\frac{\pi}{6}$
$\mathbf{45°}$	$\frac{\pi}{4}$
$\mathbf{60°}$	$\frac{\pi}{3}$
$\mathbf{90°}$	$\frac{\pi}{2}$
$\mathbf{180°}$	π
$\mathbf{270°}$	$\frac{3\pi}{2}$
$\mathbf{360°}$	2π

- Trigonometric relationships fall under the special triangles
- $x = \text{adjacent}, y = \text{opposite}, r = \text{hypotenuse}$
- Recognizing that a triangle with angles (θ) of $\frac{\pi}{4}, 45°; x = 1, y = 1, r = \sqrt{2}$
- Recognizing that a triangle with angles (θ) of $\frac{\pi}{6}, 30°; x = \sqrt{3}, y = 1, r = 2$
- Recognizing that a triangle with angles (θ) of $\frac{\pi}{3}, 60°; x = 1, y = \sqrt{3}, r = 2$

Example: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ (Exact Values)

Example: $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Example: $\sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$

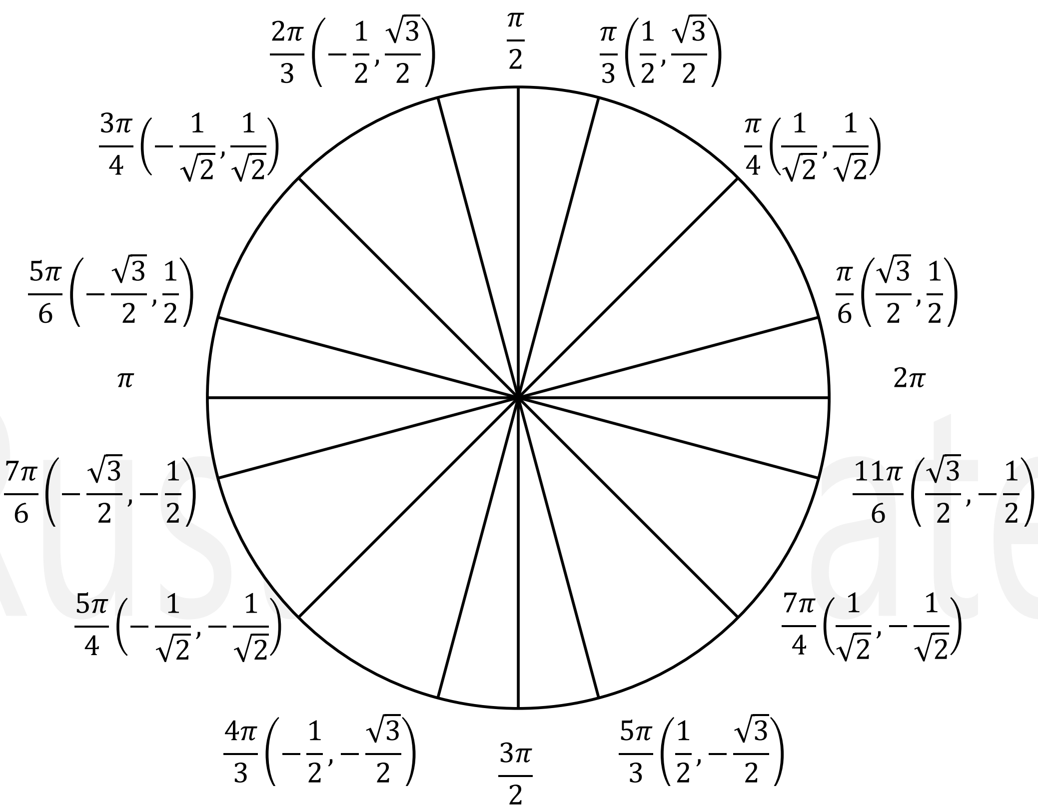
Example: $\sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = -\frac{s}{\sqrt{3}}$

Unit Circle

When the **radius** of a **unit circle** is 1, **special triangles** and relations can be drawn up between **trigonometric ratios**. The relationship between **radian** angles and side lengths of a right angle **triangle**.

- In all unit circle cases, where the radius is 1, on a Cartesian plane, the point of the terminal arm will have the coordinates x, y where $x = \cos\theta$, $y = \sin\theta$, θin standard position

Formula:



$\sin\theta = \frac{y}{r}, r = 1$

$\text{Sin}\,\theta = y$

$\cos\theta = \frac{x}{r}, r = 1$

$\cos\theta = x$

$\therefore P(x, y) = (\cos\theta, \sin\theta)$

- Evaluate for exact values only

Example: $\frac{5\pi}{3}$

$(y)\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

$(x)\cos\frac{5\pi}{3} = \frac{1}{2}$

$\tan\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

$\csc\frac{5\pi}{3} = -\frac{2}{\sqrt{3}}$

$\sec\frac{5\pi}{3} = 2$

$\cot\frac{5\pi}{3} = -\frac{1}{\sqrt{3}}$

Example: $-\frac{\pi}{4}$

$\sin -\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$\cos -\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan -\frac{\pi}{4} = -1$

$\csc -\frac{\pi}{4} = -\sqrt{2}$

$\sec -\frac{\pi}{4} = \sqrt{2}$

$\cot -\frac{\pi}{4} = -1$

Example: $\cos\frac{2\pi}{3}\cos\frac{5\pi}{6} + \sin\frac{2\pi}{3}\sin\frac{5\pi}{6}$

$= \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$

$= \frac{2\sqrt{3}}{4}$

$= \frac{\sqrt{3}}{2}$

Equivalent Trigonometric Expression

Consider the x and y values when reflected into different quadrants. In quadrant 1 the terminal arm would be $P(x, y), (\cos\theta, \sin\theta)$. Occur at $\pi, 2\pi$.

- In quadrant 1, all the ratios are positive, in quadrant 2, $\sin\theta$ is positive. In quadrant 2 the terminal arm would be $P(-x, y), (-\cos\theta, \sin\theta)$

Formula:
$$\alpha = \pi - \theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi - \theta) = \sin\theta$$

Example:
$$\sin\frac{\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}$$

Example:
$$\cos\frac{\pi}{3} = -\cos\left(\pi - \frac{\pi}{3}\right)$$
$$\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)$$

- In quadrant 1, all the ratios are positive, in quadrant 4, $\cos\theta$ is positive. In quadrant 4 the terminal arm would be $P(x, -y), (\cos\theta, -\sin\theta)$

Formula:
$$\alpha = 2\pi - \theta$$
$$\cos(2\pi - \theta) = \cos\theta$$
$$\sin(2\pi - \theta) = -\sin\theta$$

Example:
$$\cos\frac{\pi}{6} = \cos\left(2\pi - \frac{\pi}{6}\right)$$
$$\frac{\sqrt{3}}{2} = \cos\left(\frac{11\pi}{6}\right)$$
$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

- In quadrant 1, all the ratios are positive, in quadrant 3, $\tan\theta$ is positive. In quadrant 3 the terminal arm would be $P(-x, -y), (-\cos\theta, -\sin\theta)$

Formula:
$$\alpha = \pi + \theta$$
$$\cos(\pi + \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta$$

Co-related and co-functioned Identities

The following co-function identities relate. Occur at $\frac{\pi}{2}, \frac{3\pi}{2}$.

- For all occurrences of $\frac{\pi}{2}$ and the sum of the angle

Formula:

$$\cos\theta = \cos(-\theta)$$
$$-\sin\theta = \sin(-\theta)$$
$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$
$$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$
$$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$$

- For all occurrences of $\frac{\pi}{2}$ and the difference of the angle

Formula:

$$\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$$
$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$
$$\tan\theta = \cot\left(\frac{\pi}{2}-\theta\right)$$
$$\sec\theta = \csc\left(\frac{\pi}{2}-\theta\right)$$
$$\csc\theta = \sec\left(\frac{\pi}{2}-\theta\right)$$
$$\cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)$$

- For all occurrences of $\frac{3\pi}{2}$ and the sum of the angle

Formula:

$$\sin\theta = \cos\left(\frac{3\pi}{2}+\theta\right)$$
$$-\cos\theta = \sin\left(\frac{3\pi}{2}+\theta\right)$$

- For all occurrences of $\frac{3\pi}{2}$ and the difference of the angle

Formula:

$$-\sin\theta = \cos\left(\frac{3\pi}{2}-\theta\right)$$
$$-\cos\theta = \sin\left(\frac{3\pi}{2}-\theta\right)$$

- Solve for θ, Express as a function of its co-related acute angle

Example:
$$\cos\frac{\pi}{7} = \sin\theta$$
$$\frac{\pi}{7} = \frac{\pi}{2} - \theta$$
$$\theta = \frac{\pi}{2} - \frac{\pi}{7}$$
$$\theta = \frac{5\pi}{14}$$
$$\cos\frac{\pi}{7} = \cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)$$
$$\therefore \cos\frac{\pi}{7} = \sin\frac{5\pi}{14}$$

Example:
$$\cot\frac{4\pi}{9} = \tan\theta$$
$$\frac{4\pi}{9} = \frac{\pi}{2} - \theta$$
$$\theta = \frac{\pi}{2} - \frac{4\pi}{9}$$
$$\theta = \frac{\pi}{18}$$
$$\cot\frac{4\pi}{9} = \cot\left(\frac{\pi}{2} - \frac{\pi}{18}\right)$$
$$\therefore \cot\frac{4\pi}{9} = \tan\frac{\pi}{18}$$

Example:
$$\cos\frac{13\pi}{18} = -\sin\theta$$
$$\frac{13\pi}{18} = \frac{\pi}{2} + \theta$$
$$\theta = \frac{13\pi}{18} - \frac{\pi}{2}$$
$$\theta = \frac{2\pi}{9}$$
$$\cos\frac{13\pi}{18} = \cos\left(\frac{\pi}{2} = \frac{2\pi}{9}\right)$$
$$\therefore \cos\frac{13\pi}{18} = -\sin\frac{2\pi}{9}$$

Example: $\cot\frac{13\pi}{14} = -\tan\theta$

$\frac{13\pi}{14} = \frac{\pi}{2} + \theta$

$\theta = \frac{13\pi}{14} - \frac{\pi}{2}$

$\theta = \frac{3\pi}{7}$

$\cot\frac{13\pi}{14} = \cot\left(\frac{\pi}{2} + \frac{3\pi}{7}\right)$

$\therefore \cot\frac{13\pi}{14} = -\tan\frac{3\pi}{7}$

Example: $\csc a = \sec 1.45$

$1.45 = \frac{\pi}{2} - a$

$a = 1.45 - \frac{\pi}{2}$

$a = 1.45 - 1.37$

$a = 0.12$

$\csc 0.12 = \sec(1.37 - 0.12)$

$\csc 0.12 = \sec 1.45$

- Simplify the identities

Example: $\sin(\pi - x) + \cos\left(\frac{\pi}{2} + x\right) + \sin\left(\frac{3\pi}{2} - x\right) - \cos(-x)$

$= \sin x + (-\sin x) + (-\cos x) - \cos x$

$= -2\cos x$

Example: $\cos\left(\frac{\pi}{2} - x\right) - \sin(2\pi - x) - \cos(\pi - x) + \tan\left(\frac{\pi}{2} - x\right)$

$= \sin x - \sin x - \cos x + \cot x$

$= 2\sin x - \cos x + \cot x$

Compound Angle Formulas

Derived from a rotated triangle with a hypotenuse of 1

- For all sums

Formula:
$$\sin(a+b) = (\sin a)(\cos b) + (\cos a)(\sin b)$$
$$\cos(a+b) = (\cos a)(\cos b) - (\sin a)(\sin b)$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - (\tan a)(\tan b)}$$

- For all differences

Formula:
$$\sin(a-b) = (\sin a)(\cos b) - (\cos a)(\sin b)$$
$$\cos(a-b) = (\cos a)(\cos b) + (\sin a)(\sin b)$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + (\tan a)(\tan b)}$$

- Solve for the identity

Example:
$$\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{3} + \cos\tfrac{\pi}{6}\sin\tfrac{\pi}{3}$$
$$= \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$
$$= \sin\frac{\pi}{2}$$

Example:
$$\cos\tfrac{\pi}{4}\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{4}\sin\tfrac{\pi}{2}$$
$$= \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right)$$
$$= \cos\left(\frac{3\pi}{4}\right)$$

- Solve and find exact values

Example: $\sin\frac{\pi}{12} = \sin\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right)$ or $\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

- Solve through identities to get exact values

Example: Given $\sin x = -\frac{5}{12}$, in quadrant 4, $\cos y = \frac{4}{7}$, in quadrant 1

$\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$

Solve: $\sin x = -\frac{5}{12}; \frac{y}{r}$

$x = \sqrt{119}$

$\therefore \cos x = \frac{\sqrt{119}}{12}$

Solve: $\cos y = \frac{4}{7}; \frac{x}{r}$

$x = \sqrt{33}$

$\therefore \sin y = \frac{\sqrt{33}}{12}$

$\therefore\ \cos(x + y) = \left(\frac{\sqrt{119}}{12}\right)\left(\frac{4}{7}\right) - \frac{5}{12}\left(\frac{\sqrt{33}}{7}\right)$

$= \frac{4\sqrt{119} + 5\sqrt{33}}{84}$

Double Angle Formulas

Derived from doubling compound angle formulas

- For $\sin\theta$

Formula:

$$\sin 2a = \sin(a+a)$$
$$\sin 2a = \sin a\cos a + \cos a\sin a$$
$$\therefore \sin 2a = 2\sin a\cos a$$

- For $\cos\theta$

Formula:

$$\cos 2a = \cos(a+a)$$
$$\cos 2a = \cos a\cos a - \sin a\sin a$$
$$\therefore \cos 2a = \cos^2 a - \sin^2 a$$
$$\therefore \cos 2a = 1 - \sin^2 a$$
$$\therefore \cos 2a = 2\cos^2 a - 1$$

- For $\tan\theta$

Formula:

$$\tan 2a = \tan(a+a)$$
$$\tan 2a = \frac{\tan a + \tan a}{1 - \tan a\tan a}$$
$$\therefore \tan 2a = \frac{2\tan a}{1 - \tan^2 a}$$

Advanced Trigonometric Identities

To prove an **identity**, the left side and right side should be dealt individually.

- There are guidelines for proving **identities**
- Being with the more complicated side and use **identities** to transform that side
- Express everything in terms of **sine** and **cosine**
- Consider **expanding**, **factoring**, or **conjugates**
- **Quotient identities**

Formula:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$

- **Reciprocal identities**

Formula:

$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x}$$

- **Pythagorean Identities**

Formula:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

- Also recall **compound angle formulae** and **double angle formulae**
- Recall the guidelines for proving identities

Example:

$$\cos x = \frac{1}{\cos x} - \sin x \tan x$$
$$\cos x = \frac{1}{\cos x} - \sin x \frac{\sin x}{\cos x}$$
$$\cos x = \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$$
$$\cos x = \frac{1 - \sin^2 x}{\cos x}$$
$$\cos x = \frac{\cos^2 x}{\cos x}$$
$$\cos x = \cos x$$
$$\therefore L.S. = R.S.$$

Example: $1+\cos x = \frac{\sin^2 x}{1-\cos x}$

$$1+\cos x = \frac{1-\cos^2 x}{1-\cos x}$$

$$1+\cos x = \frac{(1-\cos x)(1+\cos x)}{(1-\cos x)}$$

$$1+\cos x = 1+\cos x$$

$\therefore L.S. = R.S.$

Example: $\csc x = \frac{1+\sec x}{\tan x+\sin x}$

$$\frac{1}{\sin x} = \frac{1+\frac{1}{\cos x}}{\frac{\sin x}{\cos x}+sin\ x}$$

$$\frac{1}{\sin x} = \frac{\frac{\cos x+1}{\cos x}}{\frac{\sin x+\sin x\cos x}{\cos x}}$$

$$\frac{1}{\sin x} = \frac{\cos x+1}{\cos x} \div \frac{\sin x+\sin x\cos x}{\cos x}$$

$$\frac{1}{\sin x} = \frac{\cos x+1}{\cos x} \times \frac{\cos x}{\sin x\,(1+\cos x)}$$

$$\frac{1}{\sin x} = \frac{1}{\sin x}$$

$\therefore L.S. = R.S.$

Trigonometric Functions

Calculating the **sine** and **cosine functions** in **radians**.

- $\mathbb{Z}$ is a set of integers
- The function of $y = \sin x$

Example: $y = \sin x$
Domain: $x \in [-2\pi, 2\pi]$ non-continuous, $x \in (-\infty, \infty)$ continuous
Range: $y \in [-1,1]$
Period: 2π
Symmetry: $\sin(-x) = -\sin x \rightarrow$ Odd function
x-int: $\{x \in \mathbb{R} | x = m\pi, m \in \mathbb{Z}\}$
y-int: $\{0,0\}$

- The function of $y = \cos x$

Example: $y = \cos x$
Domain: $x \in [-2\pi, 2\pi]$ non-continuous, $x \in (-\infty, \infty)$ continuous
Range: $y \in [-1,1]$
Period: 2π
Symmetry: $\cos(-x) = \cos x \rightarrow$ Even function
x-int: $\left\{x \in \mathbb{R} \middle| x = \frac{\pi}{2} + m\pi, m \in \mathbb{Z}\right\}$
y-int: $\{0,1\}$

- **Amplitude** determined by a **constant**
- Changes in **period**, result of change in distance/time for function to repeat. Determined by $\frac{2\pi}{b}$

Formula: $y = a \sin bx$

Example: $y = 4 \sin \frac{4}{3}\pi$

- **Secant** and **Cosecant** functions have specific **domain** and **range**

Example: $y = \sec x$
D: $\left\{x \in \mathbb{R} \middle| x \neq \frac{\pi}{2} + mx, m \in \mathbb{Z}\right\}$
R: $y \in (-\infty, -1] \cup [1, \infty)$

Example: $y = \csc x$
D: $\{x \in \mathbb{R} | x \neq mx, m \in \mathbb{Z}\}$
R: $y \in (-\infty, -1] \cup [1, \infty)$

Transforming Trigonometric Functions

Accompanied by standard **transformations** of **functions**, sketching **functions** can be done by addressing 5 key points in a **trigonometric function**

- The transformed **sine** and **cosine functions**

  Formula: $y = a\sin[k(x-d)] + c$

  Formula: $y = a\cos[k(x-d)] + c$

- **Vertical Stretch/Compression (Amplitude)**
  $-a$: Reflection on x-axis , $0 < a < 1$: Compression, $a > 1 =$ Stretch
- **Horizontal Stretch/Compression (Reciprocal)**
  $-k$: Reflection on y-axis , $0 < k < 1$: Stretch, $k > 1 =$ Compression
- **Phase Shift**
  $-d$:Moves Right, $+d$:Moves Left
- **Vertical Translation**
  $-c$:Moves Down, $+c$:Moves Up
- **The Period** of the **function** can be modeled by $\frac{2\pi}{k}$

Example: $y = 4\cos\left[\frac{1}{2}\left(x - \frac{3\pi}{2}\right)\right] - 1; a = 4, k = \frac{1}{2}, d = +\frac{3\pi}{4}, c = -1$

Period: $\left(\dfrac{2\pi}{\frac{1}{2}}\right) = 4\pi$

Amplitude: 4

Phase Shift: $\dfrac{3\pi}{2}$

Vertical Translation: Down 1

x	y	$x(2x + \frac{3\pi}{2})$	$y(4y - 1)$
0	1	$\frac{3\pi}{2}$	3
$\frac{\pi}{2}$	0	$\frac{5\pi}{2}$	-1
π	-1	$\frac{7\pi}{2}$	-5
$\frac{3\pi}{2}$	0	$\frac{9\pi}{2}$	-1
2π	1	$\frac{11\pi}{2}$	3

Example: $y = -2\sin\left[2\left(x + \frac{\pi}{3}\right)\right] + 2; a = 2, k = 2, d = -\frac{\pi}{3}, c = 2$

Period: $\left(\frac{2\pi}{2}\right) = \pi$

Amplitude: 2; Reflected on x-axis

Phase Shift: $\frac{\pi}{3}$

Vertical Translation: Up 2

x	y	$x(\frac{1}{2}x - \frac{\pi}{3})$	$y(-2y + 2)$
0	0	$-\frac{\pi}{3}$	2
$\frac{\pi}{2}$	1	$-\frac{\pi}{12}$	0
π	0	$\frac{\pi}{6}$	2
$\frac{3\pi}{2}$	-1	$\frac{5\pi}{12}$	4
2π	0	$\frac{2\pi}{3}$	2

- Given enough information, it is possible to determine the equation of a **trigonometric function**

Example: The average depth of water at the end of a dock is 6 feet. This varies 2 feet in both directions with the tide. Suppose there is a high tide at 4 AM. If the tide goes from low to high every 6 hours, write a cosine function $d(t)$ describing the depth (in feet) of the water as a function of time (in seconds). (note: $t = 4$ corresponds with 4 AM)

Let $d(t) = a\cos[k(t-d)] + c$

Amplitude: $a = \frac{\text{Max} - \text{Min}}{2} = \frac{8-4}{2} = 2$

Vertical Translation: $c = 6$

Period: $\frac{2\pi}{k} = 12; \therefore k = \frac{\pi}{6}$ (Horizontal Stretch)

Phase Shift: $d = 4$

Vertical Shift: $c = \frac{\text{Max} + \text{Min}}{2} = \frac{8+4}{2} = 6$

$\therefore d(t) = 2\cos[\frac{\pi}{6}(t-4)] + 6$

- Characteristics of **tangent** and **cotangent functions**
- **Tangent**, $y = \tan x$ has no **minimum** or **maximum** points. Has a **period** of π. It's **zeroes** are $\{x \in \mathbb{R} | x = m\pi, m \in \mathbb{Z}\}$. Its **vertical asymptotes** are $\{x \in \mathbb{R} | x = \frac{\pi}{2} + m\pi, m \in \mathbb{Z}\}$. Y-intercept is $(0,0)$
- **Cotangent**, $y = \cot x$ has no **minimum** or **maximum** points. Has a **period** of π. It's **zeroes** are $\{x \in \mathbb{R} | x = \frac{\pi}{2} + m\pi, m \in \mathbb{Z}\}$. Its **vertical asymptotes** are $\{x \in \mathbb{R} | x = m\pi, m \in \mathbb{Z}\}$. Y-intercept is Undefined

Solving Trigonometric Equations

Recall **trigonometric identities** and **special triangles**. Solving for both **approximate** and **exact values**.
Solve for x.

- Isolate for x and then useany method to determine the values of x
- Watch the **interval** and determine **quadrants** of x

Example: Solve for exact values for $4\cos^2 x - 3 = 0, x \in [0, 2\pi]$

$4\cos^2 x = 3$

$\cos^2 x = \frac{3}{4}$

$\cos x = \pm\frac{\sqrt{3}}{2}$

$\therefore x = \frac{\pi}{6}, \left(\pi - \frac{\pi}{6} = \frac{5\pi}{6}\right), \left(\pi + \frac{\pi}{6} = \frac{7\pi}{6}\right), \left(2\pi - \frac{\pi}{6} = \frac{11\pi}{6}\right)$

Example: Solve for exact values for $\sec^2 x - 3\sec x + 2 = 0 \; x \in [0,2\pi]$

Let $\sec x = y$

$y^2 - 3y + 2 = 0$

$(y-1)(y-2) = 0$

$(\sec x - 1)(\sec x - 2) = 0$

$\sec x = 1 \rightarrow \cos x = 1$

$\sec x = 2 \rightarrow \cos x = \frac{1}{2}$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$ (All positive quadrants of cosine)

Example: $-3\cos^2 x - 8\sin x = 0$

$-3(1 - \sin^2 x) - 8\sin x = 0$

$-3 + 3\sin^2 x - 8\sin x = 0$

$3\sin^2 x - 8\sin x - 3 = 0$

$(3\sin x + 1)(\sin x - 3) = 0$

$\sin x = 3$ (No solution)

$\sin x = -\frac{1}{3} \rightarrow x = \sin^{-1}\left(\frac{1}{3}\right) = 0.34$

$\therefore x = (\pi + 0.34 = 3.48)$

Example: Solve for exact values for $\tan\frac{x}{2}\cos^2 x - \tan\frac{x}{2} = 0$ $x \in [0,2\pi]$

$\tan\frac{x}{2}(\cos^2 x - 1) = 0$

$x = \tan^{-1} 0$

$x = \cos^{-1}(\pm 1)$

$\therefore x = 0, \pi, 2\pi$

Example: Solve for exact values for $\tan x \sin 2x - 1 = 0$ $x \in [0,\pi]$

$\left(\frac{\sin x}{\cos x}\right)(2\sin x \cos x) - 1 = 0$

$2\sin^2 x - 1 = 0$

$x = \sin^{-1}\left(\pm\frac{\sqrt{1}}{2}\right)$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$

Exponential Function

Recall law of **exponents** and its applications.

- **Exponential function** is a **base** to a **power** of x

  Formula: $y = b^x, b > 0, b \neq 0$

- Has a rate of change that is increasing/decreasing, proportional to the **function** for $b > 1/0 < b < 1$
- Domain of $\{x \in \mathbb{R}\}$
- Range of $\{y \in \mathbb{R} \mid y > 0\}$
- y-intercepts of $(0,1)$
- Horizontal asymptote at $y = 0$
- Recall all laws of **exponents**

Example:
$$y = b^1$$
$$y = 1$$

Example:
$$y = 2^{-2}$$
$$y = \frac{1}{4}$$

Example:
$$y = 2^{-3}$$
$$y = \frac{1}{8}$$

Example:
$$y = \frac{1}{2}^x$$
$$y = \frac{1}{4}$$

Example:
$$y = \frac{1}{2}^{-2}$$
$$y = 2^2$$
$$y = 4$$

Example: $\frac{(2x^{-6}y^4)^3(-4x^5y)^2}{(3y^{-7})^3(x^3y^4)}$

$\frac{(2^3x^{-18}y^{12})(-4^2x^{10}y^2)}{(3^3y^{-21})(x^3y^4)}$

$\frac{(8x^{-18}y^{12})(16x^{10}y^2)}{(27y^{-21})(x^3y^4)}$

$\frac{128x^{-8}y^{14}}{27x^3y^{-17}}$

$\frac{128x^{-11}y^{31}}{27} = \frac{128y^{31}}{27x^{11}}$

- Recall the **Absolute function**
- **Absolute function** takes the positive value of the expression
- When **graphing** an **absolute function**, only the positive values are **graphed**

Formula: $y = |x|$

Example: $y = |2 - 3|$

$y = |-1|$

$y = 1$

- Recall the **inverse function**
- $f^{-1}(x)$ is the inverse of $f(x)$. Switch x with y to find the **inverse function**. The **inverse function**, is $f(x)$ reflected on $y = x$

Formula: $x = b^y$

- Domain of $\{x \in \mathbb{R} \mid x > 0\}$
- Range of $\{y \in \mathbb{R} \mid\}$
- x-intercepts of $(1,0)$
- Vertical asymptote at $x = 0$

Example: $f(x) = 2x - 1$

$f^{-1}(x) = 2x - 1$

$y = 2x - 1$

$x = 2y - 1$

$y = \frac{x+1}{2}$

$\therefore f^{-1}(x) = \frac{x+1}{2}$

- Given a table of values, you an find the **function**, determine if its **exponential**, and determine its key features

Example:

x	y	Δy
-1	$\frac{1}{3}$	
0	1	$\frac{2}{3}$
1	3	2
2	9	6
3	27	18

$\Delta\text{y}: \frac{2}{\frac{2}{3}} = 3, \frac{6}{2} = 3; \text{y}: \frac{3}{1} = 3, \frac{9}{3} = 3$

$\because$ rate of change is increasing in proportion to the function

$\therefore$ the function is exponential

Test: $9 = b^2$

$(3)^2 = b^2$

$\because b = 3$

$\therefore y = 3^x$

D: $\{x \in \mathbb{R}\}$

R: $\{y \in \mathbb{R} \mid y > 0\}$

y-int: $(0,1)$

H.A.: $y = 0$

Logarithms

The **logarithm**, $\log x$, of a number, x, to a given base, b, is equal to the **exponent**, y, to which the based is raisedin order to produce x.

- The following are equivalent expressions

  Formula: $\log_b x = y \equiv x = b^y$

- The expression can be rewritten in both **logarithmic** form and **exponential** form

  Example: $2^3 = 8$
  $3 = \log_2 8$

  Example: $3^{-2} = \frac{1}{9}$
  $-2 = \log_3 \left(\frac{1}{9}\right)$

  Example: $\log_4 16$
  $4^x = 16$
  $4^x = 4^2$
  $x = 2$
  $\therefore 4^2 = 16$

  Example: $\log_3 \left(\frac{1}{27}\right)$
  Let: $\log_3 \left(\frac{1}{27}\right) = x$
  $3^x = \frac{1}{27}$
  $3^x = 3^{-3}$
  $\therefore x = 3$

- **Common logarithms** are **logarithms** with a base 10. They do not have to be written into the expression

  Example: $y = \log 100$
  $10^y = 100$
  $10^y = 10^2$
  $\therefore y = 2$

- The **logarithmic function** takes the form $y = \log_b x\,, b > 0, b \neq 0$
- The **logarithmic function** is also the **inverse function** of an **exponential function**. $x = b^y; y = b^x$

- **Transformations** of **exponential** and **logarithmic functions**

Formula: $y = a(b)^{k(x-d)} + c$

Formula: $f(x) = a\log_b[k(x-d)] + c$

- To graph an **exponential function**, first identify the basic **function**, then create a table of values, and lastly apply the **mapping equation**

Example: $y = -2\left(\frac{1}{2}\right)^{x-3} - 1$

$y = \frac{1^x}{2}$

$(x, y) \to (x+3, -2y-1)$

$\text{D:}\{x \in \mathbb{R}\}$

$\text{R:}\{y \in \mathbb{R} | y < -1\}$

Asymptote: $y = -1$

as $x \to \infty, y \to -1$

as $x \to -\infty, y \to -\infty$

- To graph an **logarithmic function**, first identify the basic **function**, write in **exponential form**, inverse the **function**, then create a table of values (switch x and y values), and lastly apply the **mapping equation**

Example: $y = 2\log_3[2(x-2)] + 1$

$y = \log_3 x$

$3^y = x$

$y = 3^x$

$(x, y) \to \left(\frac{1}{2}x + 2, 2y + 1\right)$

$\text{D:}\{x \in \mathbb{R} | x > 2\}$

$\text{R:}\{y \in \mathbb{R}\}$

Asymptote: $y = 2$

as $x \to \infty, y \to \infty$

as $x \to -\infty, y \to -\infty$

- **Properties** of **Logarithms**, where $x, y > 0$
- **Power Law**

Formula: $\log_{\mathrm{b}} x^n = n \log_b x$

- **Change of Base**

Formula: $\log_{\mathrm{b}} m = \frac{\log m}{\log b}$

- **Product Law**

Formula: $\log_{\mathrm{b}} x + \log_b y = \log_b(xy)$

- **Quotient Law**

Formula: $\log_{\mathrm{b}} x - \log_b y = \log_b \left(\frac{x}{y}\right)$

- **Radical Law**

Formula: $\log \sqrt[n]{x} = \log x^{\frac{1}{n}} = \frac{1}{n} \log x$

- State any **Restrictions**

Example: $\log(x+1)$
$x + 1 > 0$
$x > -1$

- Combination of these allow to evaluate **logarithms**

Example: $\log_3 \sqrt{27}$

$\log_3 (27)^{\frac{1}{2}}$

$\frac{1}{2}\log_3 27$

$\left(\frac{1}{2}\right)(3)$

$\frac{3}{2}$

Example: $\log_2 9$

$\frac{\log 9}{\log 2}$

3.17

Example: $3\log_{16} 2 + 2\log_{16} 8 - \log_{16} 2$

$\log_{16} 2^3 + \log_{16} 8^2 - \log_{16} 2$

$\log_{16} 8 + \log_{16} 64 - \log_{16} 2$

$\log_{16} \frac{(8)(64)}{2}$

$\log_{16} 256$

2

- Changing the **base** of **power** requires you to express the result in terms of a **power** with a certain **base**

Example: $8 = 2^3$

Example: $4^3 = (2^2)^3$
$= 2^6$

Example: $\sqrt{16} \times \sqrt[5]{32}^3 = 16^{\frac{1}{2}} \times 32^{\frac{3}{5}}$
$= (2^4)^{\frac{1}{2}} \times (2^5)^{\frac{3}{5}}$
$= 2^{\frac{4}{2}} \times 2^{\frac{15}{5}}$
$= 2^2 \times 2^3$
$= 2^5$

Example: 12
$2^k = 12$
$\log 2^k = \log 12$
$k \log 2 = \log 12$
$k = \frac{\log 12}{\log 2}$
$\therefore 12 = 2^{\frac{\log 12}{\log 2}}$

- Solving for **powers** with different **bases**

Example:
$$4^{2x-1} = 3^{x+2}$$
$$\log 4^{2x-1} = \log 3^{x+2}$$
$$(2x-1)\log 4 = (x+2)\log 3$$
$$2x\log 4 - \log 4 = x\log 3 + 2\log 3$$
$$2x\log 4 - x\log 3 = 2\log 3 + \log 4$$
$$x(2\log 4 - \log 3) = 2\log 3 + \log 4$$
$$x = \frac{2\log 3 + \log 4}{2\log 4 - \log 3}$$
$$x = 2.14$$

- **Extraneous roots** are invalid or non-real, because **logarithms** are positive
- Multiple methods are required to solve **exponential** and **logarithmic equations**

Example:
$$5^{3x} = 63$$
$$\log 5^{3x} = \log 63$$
$$\frac{3x\log 5}{3\log 5} = \frac{\log 63}{3\log 5}$$
$$x = 0.9$$

Example:
$$4(2^x) = 3^{x+1}$$
$$\log 4(2^x) = \log 3^{x+1}$$
$$\log 4 + \log 2^x = (\log 3)(x+1)$$
$$\log 4 + x\log 2 = x\log 3 + \log 3$$
$$x\log 2 - x\log 3 = \log 3 - \log 4$$
$$x(\log 2 - \log 3) = \log 3 - \log 4$$
$$x = \frac{\log 3 - \log 4}{\log 2 - \log 3}$$
$$\therefore x \doteq 0.7$$

Example:
$$\log_3 9 + \log_3 x = \log_3 24$$
$$\log_3 9x = \log_3 24$$
$\because$ the bases are equal
$$\frac{9x}{9} = \frac{24}{9}$$
$$\therefore x = \frac{8}{3}$$

- **Factoring** and **simplifying** may be necessary, including **quadratic** equation

Example:

$$5^{2x} - 5^x - 20 = 0$$
$$\text{Let } y = 5^x$$
$$y^2 - y - 20 = 0$$
$$(y - 5)(y + 4) = 0$$
$$y = 5, -4$$
$$5^x = 5^1 \rightarrow x = 1$$
$$5^x \neq -4 \rightarrow \text{Never}, \because \text{its an extraneous root}$$

- Conversion into **exponential** form may be necessary. Take the base of the **logarithm** to the power of the equation

Example:

$$\log 2x - \log 148 = 2$$
$$\log \frac{2x}{148} = 2$$
$$\log \frac{x}{74} = 2$$
$$\frac{x}{74} = 10^2$$
$$x = 74(100)$$
$$x = 7400$$

Example:

$$\log_3(x - 1) + \log_3(2x + 3) = 1$$
$$\log_3(x - 1)(2x + 3) = 1$$
$$\log_3(2x^2 + x - 3) = 1$$
$$2x^2 + x - 3 = 3^1$$
$$2x^2 + x - 6 = 0$$
$$(2x - 3)(x + 2) = 0$$
$$\therefore x = \frac{3}{2}, x \neq -2$$

- Consider the following properties of **logarithms**

Example: $\log_a a = 1$

Example: $\log_b b^x = x$

Example: $\log_a 1 = 0$

Example: $b^{\log_b x} = x$

Example: $\frac{1}{\log_b a} = \log_a b$

Sums and Differences of Functions

The **superposition principle** states that the **sum** of two or more **functions** can be found by adding the **ordinates** (y-coordinates) of the **function** at each **abscissa** (x-coordinate).

- **Superposition** can be **constant** or **variable**
- Given two **functions**, express as a **sum**

Example:

$h(x) = f(x) + g(x)$
When $f(x) = x^2, g(x) = 3$
$\therefore h(x) = x^2 + 3$

Example:

$h(x) = f(x) + g(x)$
When $f(x) = x^2, g(x) = x$
$h(x) = x^2 + x$
$\therefore h(x) = x(x + 1)$

- Given two **functions**, express as a **difference**

Example:

$P(n) = R(n) - C(n)$
When $P(n) = 8n, C(n) = 200 + 5n$
$P(n) = 8n - (200 + 5n)$
$P(n) = 8n - 200 - 5n$
$\therefore P(n) = 3n - 200$

Products and Quotients of Functions

Combining **functions** in these matters will draw up calculable **domain** and **range**, **intercepts**, **symmetry**, and **asymptotes**.

- Given two **functions**, express as a **product** (**expand**)

Example: $p(x) = f(x)g(x)$
When $f(x) = x + 3, g(x) = x^2 - x - 12$
$p(x) = x^3 - x^2 - 12x - 3x^2 - 3x - 36$
$\therefore p(x) = x^3 + 2x^2 - 15x - 36$

- Given two **functions**, express as a **quotient**

Example: $q(x) = \frac{f(x)}{g(x)}$
When $f(x) = x + 3, g(x) = x^2 - x - 12$
$q(x) = \dfrac{x+3}{(x+3)(x-4)}$
$\therefore q(x) = \dfrac{1}{x-4}, x \neq 4, -3$

Composite Functions

Composite functions are when given 2 **functions**, $f(x)$ and $g(x)$, the **composite** of f and g is $f \circ g(x) \equiv f(g(x))$; expressed as f of g at x

- Given 2 **functions**, determine the **composite** combinations

Example: Given $f(x) = \sqrt{x}, g(x) = x + 5$
$f \circ g(4) \equiv f(g(4))$
$= f(4 + 5)$
$= \sqrt{9}$
$= 3$

Example: Given $f(x) = \sqrt{x}, g(x) = x + 5$
$g \circ f(4) \equiv g(f(4))$
$= g(\sqrt{4})$
$= 2 + 5$
$= 7$

- Simplify **composite functions**

Example: Given $f(x) = \sqrt{x}, g(x) = x + 5$
$f \circ g(x) \equiv f(g(x))$
$= f(x + 5)$
$= \sqrt{x + 5}$

Example: Given $f(x) = \sqrt{x}, g(x) = x + 5$
$g \circ f(x) \equiv g(f(x))$
$= g(\sqrt{x})$
$= \sqrt{x} + 5$

Example: Given $f(x) = \sqrt{x}, g(x) = x + 5$
$g \circ g(x) \equiv g(g(x))$
$= g(x + 5)$
$= x + 10$

- Because $g \circ f(x) \neq f \circ g(x)$; therefore, **composition** is not **commutative**

- The only time **composition** is **commutative** is when the **composition** is with itself and its **inverse**

Example: Given $f(x) = \frac{3}{x-4}, g(x) = x^2$; Determine the domain

$$f \circ g(x) \equiv f(g(x))$$
$$= f(x^2)$$
$$= \frac{3}{x^2 - 4}, x \neq \pm 2$$
$$x \in (-\infty, -2) \cup (-2,2) \cup (2, \infty)$$
$$y \in (-\infty, 0) \cup (0, \infty)$$

Example: Given $f(x) = \frac{3}{x-4}$

$$f^{-1}(x)$$
$$y = \frac{3}{x - 4}$$
$$x = \frac{3}{y - 4}$$
$$y - 4 = \frac{3}{x}$$
$$y = \frac{3}{x} + 4 \text{ or } y = \frac{3 + 4x}{x}$$

Example: Given $f(x) = \frac{3}{x-4}, g(x) = x^2$

$$f \circ f^{-1}(x) \equiv f(f^{-1}(x))$$
$$= f\left(\frac{3 + 4x}{x}\right)$$
$$= \left(\frac{3}{\frac{3 + 4x}{x}} - 4\right)$$
$$= \left(\frac{3}{\frac{3 + 4x - 4x}{x}}\right)$$
$$= \frac{3}{\frac{3}{x}}$$
$$= x$$

Example: Given $f(x) = \frac{3}{x-4}, g(x) = x^2$

$$f^{-1} \circ f(x) \equiv f^{-1}(f(x))$$
$$= x$$

Rate of Change

Average rate of change is a change that takes place over an **interval**

- **Quotient** of change in y and x

Formula: $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

- Given an **interval**, the **average rate of change** can be found

Formula:
$$a \leq x \leq b$$
$$\frac{f(b) - f(a)}{b - a}$$

Example:
$$f(x) = x^2$$
$$2 \leq x \leq 3$$
$$\frac{f(3) - f(2)}{3 - 2} = \frac{9 - 4}{1}$$

- A **secant** is a **line** joining 2 **points** on any **curve**
- **Instantaneous rate of change** is the **slope** of a **tangent** to a **point** on any **curve**
- There are multiple methods to find the **instantaneous rate of change**
- First method: average of **average rate of change** requires two **intervals** around a set value. An interval before the value and one after

Example:
$$f(x) = x^2\ ; x = 2$$
$$1 \leq x \leq 2 = 3$$
$$2 \leq x \leq 3 = 5$$
$$\frac{3 + 5}{2} = 4$$

- The second method requires **graphing** the **function**, then drawing the **tangent** and picking 2 points off the **tangent** line to calculate the **slope**
- The third method involves analyzing all **secants** close to the value

Example: $f(x) = x^2\ ; x = 2$

$P(2, f(2)) = P(2,4)$

$Q(x, f(x))$

$$m_{PQ} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 4}{x - 2}$$

$x \to 2^-$	m_{PQ}	$x \to 2^+$	m_{PQ}
1.9	3.9	2.1	4.1
1.99	3.99	2.01	4.01
1.999	3.999	2.001	4.001

Increasing and Decreasing Functions

Methods for identifying different types of **functions** and for what **intervals** they are increasing or decreasing.

- A **function** f is increasing on an **interval** (a, b) is $f(x_2) > f(x_1)$ when $x_2 > x_1$ for all $x_i \in (a, b)$
- A **function** f is decreasing on an **interval** (a, b) is $f(x_2) < f(x_1)$ when $x_2 > x_1$ for all $x_i \in (a, b)$
- Determine the turning points in a function and asymptotes. Express increasing and decreasing areas in interval notation omitting turning points and asymptotes

Example: $f(x) = x^3 - 2x$

Local Max: $(-2,16)$

Local Min: $(2, -16)$

$\therefore x \in (-\infty, -2) \cup (2, \infty)$

Calculus

Limits

A **limit** is the value a **function** approaches as the x-value approaches a number. **Limits** are **behaviours** for when x approaches a value

- A **function** has a limit L as $x \to a$

  Formula: $\lim_{x\to a} f(x) = L$

- Provided that value of $f(x)$ gets closer to L as x gets closer to a on both sides of a, $a^{\pm}$

  Example: $f(x) = \frac{x-3}{x^2-4x+3} = \frac{x-3}{(x-3)(x-1)} = \frac{1}{x-1}, x \neq 3, 1$

  $\lim\limits_{x\to 3^-} f(x) = 0.5$

  $\lim\limits_{x\to 3^+} f(x) = 0.5$

  $\therefore \lim\limits_{x\to 3} f(x) = 0.5$

- A **limit** exists if and only if both of its one-sided **limits** exist and are equal

  Example: $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = L$

- A **limit** exists as $x \to a$ (approaches), not equal $x = a$

  Example: $\lim_{x\to a} f(x) \neq f(a)$

- Multiple cases for limits including do not exist (DNE)

  Example: $f(x) = \frac{1}{x^2}$

  $\lim\limits_{x\to 0} f(x) = \text{DNE}$

Properties of Limits

- For any **constant function** c and any real number a

  Formula: $\lim_{x \to a} C = C$

- For any **function** $f(x)$ and any real number a

  Formula: $\lim_{x \to a} x = a$

- For any 2 or more **functions** that have an existing limit with a **constant**¸ several rules apply

  Formula: $\lim_{x \to a} Cf(x) = C \lim_{x \to a} f(x)$

  Formula: $\lim_{x \to a}[f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

  Formula: $\lim_{x \to a}[f(x) \times g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$

  Formula: $\lim_{x \to a}[f(x) \div g(x)] = \lim\limits_{x \to a} f(x) \div \lim\limits_{x \to a} g(x), \lim\limits_{x \to a} g(x) \neq 0$

  Formula: $\lim_{x \to a}[f(x)]^2 = \lim_{x \to a}[f(x) \times f(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} f(x)$
  $= \left[\lim\limits_{x \to a} f(x)\right]^2$

  Formula: Let $f(x) = x$
  $\lim\limits_{x \to a} f(x)^n = \lim\limits_{x \to a} x^n = \left[\lim\limits_{x \to a} f(x)\right]^n = a^n$

- For any **polynomial function**, **factor** and cancel, **rationalize**, then substitute

  Formula: $\lim_{x \to a} P(x) = P(a)$

  Example: $\lim_{x \to 2}(3x^2 + 5x - 4)$
  $= \lim\limits_{x \to 2}(3x^2) + \lim\limits_{x \to 2}(5x) - \lim\limits_{x \to 2} 4$
  $= 3\lim\limits_{x \to 2} x^2 + 5\lim\limits_{x \to 2} x - 4$
  $= 3\left(\lim\limits_{x \to 2} x\right)^2 + 5\left(\lim\limits_{x \to 2} x\right) - 4$
  $= 3(2)^2 + 5(2) - 4$
  $= 18$

- Like **polynomial functions**, **rational functions** need to be **factored** and cancel in order to justify **restrictions** and the **asymptotes**. Disregard **restrictions** because **limits** solve for approaching value

Example: $\lim_{x\to 3}\left(\frac{x^2-x-6}{x-3}\right)$

$=\frac{\lim\limits_{x\to 3}(x^2-x-6)}{\lim\limits_{x\to 3}(x-3)}$

$=\frac{3^2-3-6}{3-3}$

$=\frac{0}{0}$

$=0$ improper solved

Example: $\lim_{x\to 3}\left(\frac{x^2-x-6}{x-3}\right)$

$=\frac{\lim\limits_{x\to 3}(x-3)(x+2)}{\lim\limits_{x\to 3}(x-3)}$

$=\lim\limits_{x\to 3}(x+2)$

$=3+2$

$=5$

Example: $\lim_{x\to 1}\frac{x^2-2x-8}{x+1}$

$=\lim_{x\to 1}\frac{(x+2)(x-4)}{x+1}$

Cannot solve, $\because x=1, \lim_{x\to 1}f(x)=DNE$

- **Radical functions** require **rationalizing** and must be considered from both left and right sides in order to form an appropriate **limit**

Example:

$$\lim_{x\to 2}\sqrt{x-2}$$
$$=\lim_{x\to 2}\sqrt{2\times 2}$$
$$=\sqrt{0}$$
$$=0 \text{ improper solve}$$
$$\lim_{x\to 2^+}=0$$
$$\lim_{x\to 2^-}=DNE$$
$$\therefore \lim_{x\to 2}=DNE$$

Example:

$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$$
$$=\lim_{x\to 0}\left[\frac{\sqrt{x+3}-\sqrt{3}}{x}\times\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right]$$
$$=\lim_{x\to 0}\frac{x+3-3}{x\left(\sqrt{x+3}+\sqrt{3}\right)}$$
$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+3}+\sqrt{3}\right)}$$
$$=\lim_{x\to 0}\frac{1}{\left(\sqrt{x+3}+\sqrt{3}\right)}$$
$$=\frac{1}{2\sqrt{3}}$$
$$=\frac{\sqrt{3}}{6}$$

Continuity

A **continuous function** is a **function** that does not stop or have any breaks in the **function**.

- **Continuous functions** include **linear**, **polynomial**, and **sinusoidal functions**
- **Discontinuous functions** include **rational functions** (**asymptotic** or **hole**) and **jump-discontinuity** or **piecewise functions**
- Several definitions apply to a **continuous function**
  - $f(a)$ is defined
  - $\lim_{x\to a} f(x)$ is defined
  - $\lim_{x\to a} f(x) = f(a)$
- Rules
  - All **polynomial**, **exponential**, and **sinusoidal functions** are **continuous** infinitely, $x \in \mathbb{R}$
  - All **radical** ($\sqrt[n]{x}, n$ is even) and all $\log x$ **functions** are **continuous** for $x > 0$
  - All **radical** ($\sqrt[n]{x}, n$ is odd) are **continuous** for all $x \in \mathbb{R}$
  - All **rational functions** are made up of **continuous polynomial functions** and therefore **continue** everywhere except for **restriction** in the **denominator**
  - f and g are **continuous** at $x = a$, the following apply: $(f \pm g)$ is **continuous** at $x = a$, (fg) is **continuous** at $x = a$, and $\left(\frac{f}{g}\right)$ is **continuous** at $x = a, g(a) \neq 0$
- In order to remove a **discontinuity**, a **function** that has a **hole** in the **graph** needs a **point**; therefore redefine a **hole function** as a **piecewise function** including a **point**

  Example: $f(x) = \frac{2x^2-5x-3}{x-3} = \frac{(2x+1)(x-3)}{x-3} = 2x+1, \text{hole at } x = 3$

  $\lim\limits_{x\to 3} f(2x+1) = 7, \therefore P(3,7)$

  $$\therefore f(x) = \begin{cases} \dfrac{2x^2-5x-3}{x-3}, & x \neq 0 \\ 7, & x = 3 \end{cases}$$

- A **jump-discontinuous function** is a **piecewise function**
- A **infinite discontinuous function** is a **rational function** with a **vertical asymptote(s)**
- A **removable discontinuous function** is a **rational function** with a **hole**

Limits involved Infinity

Vertical asymptotes occur on the y-axis due to **restrictions**. **Horizontal asymptotes** occur on the x-axis due to **end behaviour**. Opposed to the traditional **table of values** method to prove and justify the **equation** of the **asymptotes**, **limits** are an alternative method.

Vertical Asymptotes

- Consider a simple **rational function**

Example: $f(x) = \frac{1}{x}, \lim_{x\to 0} \frac{1}{x}$

As $x \to 0^-$, $\lim\limits_{x\to 0^-} f(x) = -\infty$

As $x \to 0^+$, $\lim\limits_{x\to 0^+} f(x) = \infty$

$\therefore \lim\limits_{x\to 0} f(x) = DNE$

Example: $f(x) = \frac{1}{x^2}$

$\therefore \lim\limits_{x\to 0} f(x) = \infty$

- Calculating **limits** at **restrictions** requires the use of identifying the **restriction** (**denominator**) **factor(s)** and then finding its **limit** from both the left and right sides. Provided the **limit** is DNE, then you have proved the **vertical asymptote**
- Find values **approaching** a from the **left** and **right** to see if the result is positive/negative infinity

Example: $f(x) = \frac{3x}{x-2}, x \neq 2$

$$\lim_{x\to 2^-} \frac{3x}{x-2} = -\infty$$

$$\lim_{x\to 2^+} \frac{3x}{x-2} = \infty$$

$$\because \lim_{x\to 2} \frac{3x}{x-2} = DNE$$

$$\therefore V.A.: x = 2$$

- Working with **trigonometric functions** and **limits**

Example: $\lim_{x\to\frac{\pi}{2}} \tan x$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty$$

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\sin x}{\cos x} = -\infty$$

$$\therefore \lim_{x\to\frac{\pi}{2}} \tan x = DNE, \therefore V.A.: x = \frac{\pi}{2}$$

Horizontal Asymptotes

- Consider a simple **rational function**

Example: $f(x) = \frac{1}{x}$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\therefore H.A.: y = 0$$

Formula: $\frac{1}{x^n}$ as $x \to \pm\infty = 0$

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

- Calculating end behaviour requires that you force the identity $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ into any function in order to avoid $\frac{\pm\infty}{\pm\infty} = 1$

Example: $f(x) = \frac{3x+2}{4x-1}$

$$\lim_{x\to\infty} \frac{3x+2}{4x-1} = \frac{3x+2}{4x-1}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) = \frac{\left(3+\frac{2}{x}\right)}{\left(4-\left(\frac{1}{x}\right)\right)} = \frac{3+0}{4+0} = \frac{3}{4}$$

$$\therefore H.A.: y = \frac{3}{4}$$

Example: $f(x) = \frac{3x+2}{x^2-4}$

$$\lim_{x\to\infty} \frac{3x+2}{x^2-4} = \frac{3x+2}{x^2-4}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \left(\frac{\frac{3}{x}+\frac{2}{x^2}}{1-\frac{4}{x^2}}\right) = \frac{0+0}{1+0} = 0$$

$$\therefore H.A.: y = 0$$

Derivatives

A **derivative** is the **slope** of a **tangent line** on any **curve**, also recognized as the **instantaneous rate of change** at any **point** on a **curve**. Calculated using **first principles**.

- **Slope** is calculated on 2 **points** off a **curve**. The **line** going through the **points** is called a **secant**
- **Slope** of the **secant** represents **average rate of change** of a **function** over the **interval** $a \leq x \leq b$
- Instantaneous rate of change is the slop of the tangent, with an interval of 0 $(a \leq x \leq b)$

Formula: $m_{ab} = \frac{f(b)-f(a)}{b-a}$

- **Instantaneous rate of change** is the **slope** of the **tangent**, with an **interval** of 0 $(a \leq x \leq b)$
- Let the **denominator** or interval be h, and the **slope** of the **secant approaches** the **slope** of the **tangent** as the size of the **interval approaches** 0. Use **first principles formula** with any notation (multiple notations; prime)

Formula: $f'(x) = \lim_{h\to 0} \frac{f(x+b)-f(x)}{h}, x = a$

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = y' = m_{tangent}$$

- The result of the **formula** will be the **derivative** of the **formula**, only works where a **limit** exists
- The **derivative** of any **polynomial** will be a **degree** less than the original **function**

Example: $f(x) = 4x^2 - 3x + 5$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{4(x+h)^2 - 3(x+h) + 5 - (4x^2 - 3x + 5)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{4x^2 + 8xh + 4h^2 - 3x - 3h + 5 - 4x^2 + 3x - 5}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{8xh + 4h^2 - 3h}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{h(8x + 4h - 3)}{h}$$
$$f'(x) = \lim_{h\to 0} 8x + 4h - 3$$
$$f'(x) = 8x + 3$$
$$f'(1) = 5$$

Differentiability

A **function** $f(x)$ is **differentiable** at $x = a$ if $f'(a)$ exists. **Differentiability** is the ability to find the **slopes** of a **tangent** at $x = a$. To **differentiate** is to find the **derivative**. A **function** must be **continuous** and smooth in order to be **differentiable** for all $x = a$

- All **polynomial**, **sinusoidal**, and **exponential functions** are **differentiable** everywhere for all of $x = a$
- All **logarithms** and even **radical functions** are **differentiable** for $x > 0$
- All odd **radical functions** are **differentiable** for all except $x = 0$

Example: $f(x) = \sqrt{x}, x = 0$

$$f'(0) = \lim_{h \to 0}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{\sqrt{0+h} - \sqrt{0}}{h}\right)\left(\frac{\sqrt{0+h} + \sqrt{0}}{\sqrt{0+h} + \sqrt{0}}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{\sqrt{h}}{h}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{\sqrt{h}}{h}\right)\left(\frac{\sqrt{h}}{\sqrt{h}}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{h}{h\sqrt{h}}\right)$$

$$f'(0) = \lim_{h \to 0}\left(\frac{1}{\sqrt{h}}\right)$$

$$f'(0) = \frac{1}{\sqrt{0}}$$

$$f'(0) = DNE$$

The Constant and Power Rule

Constant Rule

- **Constant functions** have the form $f(x) = c$ and a graph of a **horizontal line**
- Prove the **constant** rule using **first principles**

Example:

$$f(x) = C$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{C - C}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{0}{h}$$
$$f'(x) = 0$$
$$\therefore \frac{d}{dx} C = 0, f(x) = C, f'(x) = 0$$

Power Rule

- For **linear functions** the **derivative** is 1

Example:

$$f(x) = x$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{x + h + x}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{h}{h}$$
$$f'(x) = \lim_{h \to 0} 1$$
$$f'(x) = 1$$
$$\therefore \frac{dx}{dx} = 1, f(x) = x, f'(x) = 1$$

- The **constant** and **linear** rules are **special cases** of the power rule $y = C \rightarrow y = Cx^0, y = x \rightarrow y = x^1$

- **Power rule** can be proven through **first principles** for a **power function** (use the **binomial expansion theorem**)

Example: $\frac{d}{dx}x^n$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\left(\left(x^n - nx^{n-1}h + \left(\frac{n(n-1)}{h}\right)x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n\right)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\left(nx^{n-1}h + \left(\frac{n(n-1)}{h}\right)x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right)}{h}$$

$$f'(x) = nx^{n-1}$$

$$\therefore \frac{d}{dx}x^n = nx^{n-1}, f(x) = x^n, f'(x) = nx^{n-1}$$

Example: $f(x) = x^3$

$$f'(x) = 3x^2$$

Example: $f(x) = x^8$

$$f'(x) = 8x^7$$

Example: $f(x) = \frac{1}{x^5}$

$$f'(x) = x^{-5} = -5x^{-6} = -\frac{5}{x^6}$$

Example: $f(x) = x^{\frac{3}{2}}$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

Example: $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

- With a **constant**, use the **power** and **limit rules** to prove

Example: $f(x) = 3x^2$

$\frac{d}{dx}\left(Cf(x)\right)$, Let $g(x) = Cf(x)$

$f'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{Cf(x+h) - Cf(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{C\left(f(x+h) - f(x)\right)}{h}$

$f'(x) = \lim_{h\to 0} C \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h\to 0} C \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = Cf'(x)$

Example: $f(x) = \frac{1}{-2\sqrt[5]{x^3}} = -\frac{1}{2}\left(x^{-\frac{3}{5}}\right)$

$f'(x) = \frac{3}{10}\left(x^{-\frac{8}{5}}\right)$

$f'(x) = \frac{3}{10\sqrt[5]{x^4}}$

- All formulas for **constant** and **power rules**

Formula: $f(x) = C$

$f'(x) = 0$

Formula: $f(x) = x$

$f'(x) = 1$

Formula: $f(x) = x^n$

$f'(x) = nx^{n-1}$

Formula: $f(x) = Cx^n$

$f'(x) = C(nx^{n-1})$

The Sum, Difference, and Polynomial Rules

Recall **polynomial functions** are made by the addition and subtraction of individual **terms** and each **term** is its own **function**. The **derivative** of each individual **function** is the **derivative** of the whole **polynomial function**.

- The sum and difference rule can be proven using **first principles**

  Formula: $p(x) = h(x) \pm k(x)$
  $p'(x) = h'(x) \pm k'(x)$

- A **polynomial function** is the addition or subtraction of 2 or more **power functions**

  Example: $f(x) = -3x^5 + 4x^2 - 3\sqrt{x}$
  $f(x) = -3x^5 + 4x^2 - 3x^{\frac{1}{2}}$
  $f'(x) = -15x^4 + 8x - \frac{3}{2}x^{-\left(\frac{1}{2}\right)}$
  $f'(x) = -15x^4 + 8x - \frac{3}{2\sqrt{x}}$

Velocity and Acceleration

Velocity is the **slope** of a distance time graph, thus a **derivative** of a distance time graph. **Acceleration** is the slope of a **velocity** time graph, thus the **derivative** of a velocity time graph.

- Second **derivatives** is when you take the **derivative** of an already **derived function**

  Formula: $a(t) = v'(t) = d''(t)$

The Product Rule

The **derivative** of a **product** is not the **product** of its **derivatives**. Prove using **first principles**.

- **Expanding** the **function** initially also works
- Of $f(x) = g(x)k(x)$ and $g(x)$ and $k(x)$ are differentiable

Formula: $f'(x) = g(x)k'(x) + k(x)g'(x)$

$$\frac{d}{dx}[g(x)k(x)] = k(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}k(x)$$

The Chain Rule

Addresses the **composite function** in the form $f(g(x))$ where f and g are **differentiable**

- Use **first principles** to prove

Formula: $\frac{d}{dx}[F(x)] = \frac{d}{dg(x)} f\left(g(x)\right)\left(\frac{dg(x)}{dx}\right)$

$f'\left(g(x)\right)g'(x)$

Intervals of Increase and Decrease

Recall that **intervals** can be used to determine areas of **increase** and **decrease** within a **function** along with local **minimum** and **maximums**.

- If a **function** is **increasing**, the **slope** or Δy is positive
- If a **function** is **decreasing**, the **slope** or Δy is negative
- Δx is always positive (left to right)
- Test for **increase** and **decrease** in **functions** by taking the **derivative** of a **function**
    - If $f'(x) > 0$ for all x of an **interval**, then $f(x)$ is increasing for that **interval**
    - If $f'(x) < 0$ for all x of an **interval**, then $f(x)$ is decreasing for that **interval**

Example: $f(x) = \frac{2}{3}x^3 - 2x^2 + 16x + 1$

$f'(x) = -2(x^2 - 2x + 8)$

$f'(x) = -2(x + 4)(x - 2)$

$f'(x) = 0, x = -4, 2$ Can represent a max or min

Interval/$f'(x)$	$(-\infty, -4)$	$(-4, -2)$	$(2, \infty)$
$\mathbf{-2}$	$-$	$-$	$-$
$\mathbf{(x + 4)}$	$-$	$+$	$+$
$\mathbf{(x - 2)}$	$-$	$-$	$+$
Sign of $f'(x)$	$-$	$+$	$+$
Behaviour of $f(x)$	Decrease	Increase	Decrease

$\therefore f(x)$ is increasing on $(-4, -2)$

$f(x)$ is decreasing on $(-\infty, -4) \cup (2, \infty)$

Local minimum at $x = -4$, local maximum at $x = 2$

Minimums, Maximums, and the First Derivative

Minimums and **maximums** can be **local** or **absolute**.

- A **function** $f(x)$ has a **local maximum** (or **minimum**) at C if $f(C) \geq f(x)$ (or $f(X) \leq f(x)$) for all x close to C
- A **function** $f(x)$ has an **absolute maximum** (or **minimum**) at C if $f(C) \geq f(x)$ (or $f(X) \leq f(x)$) for all x in the **domain** of $f(x)$
- A **maximum** or **minimum** is when the **slope** is $0, f'(x) = 0$
- Not all $f'(x) = 0$ are **maximums** or **minimums** (turning points)
- **Critical numbers** are **points** on the **graph** where $f'(C) = 0$ or $f'(C) = DNE$
- **Critical numbers** are when things are changing on the **graph** or something different occurs
- First derivatives test for **local/absolute extrema**. Let C be a **critical number** of **function** that is **continuous** over a **given interval**
    - If $f(x)$ changes from **negative** to **positive** at C, then the **point** $(C, f(C))$ is a **minimum**
    - If $f(x)$ changes from **positive** to **negative** at C, then the **point** $(C, f(C))$ is a **maximum**
    - If $f'(x)$ does not change signs then $f(C)$ is not a **maximum** or **minimum**
    - If $f'(x)$ is **negative** for all $x < C$ and $f'(x)$ is **positive** for all $x > C$, then $f(C)$ is an **absolute minimum**
    - If $f'(x)$ is **positive** for all $x < C$ and $f'(x)$ is **negative** for all $x > C$, then $f(C)$ is an **absolute maximum**

Inflection Point, Concavity, and the Second Derivative

The **second derivative** of a **function** reveals the **point** of **inflection** and **concavity**.

- **Concavity** is the **curvature** (shape) of the **graph**
- **Curvature** depends on a change in **slope**
- If the ends of the graph point up, then **curvature** is **concave** up
- If the ends of the graph point down, then **curvature** is **concave** down
- $f(x)$ is **concave** up if $f'(x)$ is increasing
- $f(x)$ is **concave** down if $f'(x)$ is decreasing
- For a **differentiable function** where a **second derivative** exists:
  - $f(x)$ is **concave** up if $f''(x) > 0; +$
  - $f(x)$ is **concave** down if $f''(x) < 0;\ -$
- A **point** of **inflection** is a **point** on the **graph** where **curvature** changes from **concave** up to down (vice versa) if $(C, f(C))$ is an **inflection** point then $f''(x) = 0$ provided $f''(C)$ exists

Oblique Asymptotes

The **end behaviour** for a **rational function**. Is a slant **asymptote** if the vertical distance between the curve $y = f(x)$ and the slanted line approaches 0 as $\rightarrow \infty$.

- Occurs when the **numerator** is a **degree** less than the **denominator**

Formula: $\lim_{x \to \infty}[f(x) - (mx + b)] = 0$

- If $f(x) = \frac{1}{x^n}, H.A.: y = 0$
- If $f(x) = \frac{ax^n}{bx^n}, H.A.: y = \frac{a}{b}$
- If $f(x) = \frac{x^n}{x^{n-1}}, H.A.: y = mx + b$

Example: $y = \frac{x^2-x-6}{x-1}$

$$y = x - \frac{6}{x-1}$$

- The **quotient** is the **oblique asymptote**

Curve Sketching

Using the original **function** and the **first** and **second derivatives**, it is possible to sketch a **graph**.

- Working with $f(x)$
  - Factor
  - Find zeroes (x-intercepts)
  - Find y-intercept
  - End behaviour
  - Restrictions
- Working with $f'(x)$
  - Factor
  - $f'(x) = 0$ are critical numbers
  - Maximum and minimum in a behaviour chart
- Working with $f''(x)$
  - Factor
  - $f''(x) = 0$ are possible inflections
  - Inflection/Curvature in behaviour chart

Example: $f(x) = \frac{x^3}{x^2-4}$

$$f(x) = \frac{x^3}{(x-2)(x+2)}$$

$x = 0, x \neq \pm 2$

$V.A.: x = \pm 2$

y-int: $f(0) = 0$

End Behaviour: Oblique

$$f(x) = x + \frac{4x}{x^2-4}$$

$O.A.: y = x$

$$f'(x) = \frac{x^4 - 12x^2}{(x^2-4)^2}$$

$$f'(x) = \frac{x^2(x^2-12)}{(x^2-4)^2}$$

$$f'(x) = \frac{x^2(x-\sqrt{12})(x+\sqrt{12})}{(x^2-4)^2}$$

Critical Numbers: $f'(x) = 0$

$x = \pm\sqrt{12}, 0, x \neq \pm 2$

$x = \pm 2$

$f(\pm\sqrt{12}) = 5.2(\text{min}), -5.2(\text{max})$ Local

$$f''(x) = \frac{8x(x^2+12)}{(x-2)^3(x+2)^3}$$

$f''(0) = 0, V.A.: x = \pm 2; f(0) =$ Inflection

Limits of Trigonometric Functions

Solve **trigonometric functions** using several methods including **table of values**, **substitution**, **factor** and **rationalizing**, and **squeeze theorem**.

- Solving **limits** using **substitution**

Example: $\lim_{x\to\pi} \cos x$

$= \cos\pi$

$= 1$

Example: $\lim_{x\to\frac{\pi}{4}} (\sin x - \cos x)$

$= \sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= 0$

- Solving **limits** through **factoring**

Example: $\lim_{x\to\frac{\pi}{2}} \left(\frac{\sin^2 x + \cot x - 1}{\cos x}\right)$

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{(1-\cos^2 x) + \cot x - 1}{\cos x}\right)$$

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{(-\cos^2 x) + \cot x}{\cos x}\right)$$

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{-\cos^2 x + \frac{\cos x}{\sin x}}{\cos x}\right)$$

$$\lim_{x\to\frac{\pi}{2}} -\cos x + \frac{1}{\sin x}$$

$$= -\cos\frac{\pi}{2} + \frac{1}{\sin\frac{\pi}{2}}$$

$$= 1$$

- Solving **limits** through **rationalizing**

Example: $\lim_{x\to0}\left(\frac{\sin x}{\sqrt{\sin x}}\right)$

$$\lim_{x\to0}\left(\frac{\sin x}{\sqrt{\sin x}}\times\frac{\sqrt{\sin x}}{\sqrt{\sin x}}\right)$$
$$\lim_{x\to0}\left(\frac{\sin x\left(\sqrt{\sin x}\right)}{\sqrt{\sin x}}\right)$$
$$=\sqrt{\sin 0}$$
$$=0$$

- Unable to **factor**, **rationalize**, or **substitute**, use **squeeze theorem**
  - $f(x)\le g(x)\le h(x)$ for all x then $\lim_{x\to a} f(x)=\lim_{\mathrm{x\to a}} h(x)=L$
  - $\therefore \lim_{x\to a} g(x)=L$

Example: $\lim_{x\to0} x^2\cos\left(\frac{1}{x}\right)$

$\cos a$ has a **minimum** and **maximum** of 1
$$-1\le\cos a\le1$$
$$-1\le\cos\frac{1}{x}\le1$$
$$-x^2\le x^2\cos\frac{1}{x}\le x^2$$
$$\lim_{x\to0}(-x^2)=0$$
$$\lim_{x\to0}(x^2)=0$$
$$\therefore L.S.\text{ and }R.S.=0,\therefore\lim_{x\to0}x^2\cos\left(\frac{1}{x}\right)=0$$

- Definition: $\lim_{x\to0}\frac{\sin x}{x}=1$

Derivatives of Trigonometric Functions

All **derivative** rules apply to **trigonometric functions**

- Definition: $\lim_{x\to 0}\frac{\sin x}{x} = 1$
- Definition: $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$
- **First principles** can solve **trigonometric functions**

Example:

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f''''(x) = \sin x \ldots$$

- Apply derivative rules

Example:

$$f(x) = 3\sin x + 2x^2$$
$$f'(x) = 3\cos x + 4x$$

Example:

$$g(x) = \sin x \cos x$$
$$g(x) = \cos x \cos x - \sin x \sin x$$
$$g'(x) = \cos^2 x - \sin^2 x$$
$$g'(x) = \cos 2x$$

Example:

$$h(x) = \sin\sqrt{x} = \sin x^{\frac{1}{2}}$$
$$h'(x) = \left(\cos x^{\frac{1}{2}}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$h'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Example:

$$m(x) = \sqrt{(\cos(x^2 + 4x))} = \cos(x^2 + 4x)^{\frac{1}{2}}$$
$$m'(x) = \frac{1}{2}(\cos(x^2 + 4x))^{-\frac{1}{2}}(-\sin(x^2 + 4x))(2x + 4)$$
$$m'(x) = \frac{-(x + 2)\sin(x^2 + 4x)}{\sqrt{\cos(x^2 + 4x}}$$

Derivatives of Exponential Functions

The **derivative** of any **exponential function** is an **exponential function** multiplied with a **constant**.

- If $f(x) = a^x, f'(x) = [f'(0)][f(x)]$
- **First principles** can solve **exponential functions**

Example: $f(x) = a^x$
$f'(x) = a^x C$

- If $y = 2^x$ the **derivative** is below $f(x)$, **compression**
  - $\lim_{h \to 0} \left(\frac{a^h - 1}{h}\right) < 1$
- If $y = 3^x$ the **derivative** is above $f(x)$, **expansion**
  - $\lim_{h \to 0} \left(\frac{a^h - 1}{h}\right) > 1$
- The **base** of the **exponential function** between 2 and 3 will have a **derivative** the same as the **original function**
- $e \cong 2.718 \ldots$
- $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
- $1 = \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)$

Example: $f(x) = e^{2x}$
$f'(x) = 2e^{2x}$

Example: $f(x) = 2e^{-x}$
$f'(x) = -2e^{-x}$

Example: $f(x) = e^{1-2x}$
$f'(x) = -2e^{1-2x}$

The Natural Logarithm

The **natural logarithm** has a base e and is written as a **lawn function**.

- An **exponentials inverse** is a **logarithmic function** and vice-versa

Formula: $\log_e x = \ln x$

- The **lawn function** can be used in place of a **logarithm function** because a **logarithm function** has a **base** 10

Example: $y = e^x$
$y = \ln x$

Example: $\ln e^x$
$= x \ln e$
$= x(1)$
$= x$

Example: $e^{\ln x}$
$\ln a = \ln x$
$a = x$
$e^{\ln x} = x$

- e and $\ln$ cancel each other out because they are **inverse functions**

Example: $e^x = 7$
$\ln e^x = \ln 7$
$x = \ln 7$

Example: $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

Example: $\ln(5x - 2) = 4$
$e^{\ln(5x-2)} = e^4$
$5x - 2 = e^4$
$x = \dfrac{e^4 + 2}{5}$

Derivatives of Exponential Functions

A pattern can be found when looking for the **coefficient** values. The pattern follows the **lawn function**.

- The **constant** depends on the **base** and will be different for each **exponential function**
- The value of the **constant** is $f'(0)$

Formula: $f(x) = a^x$

$f'(x) = \ln a \cdot a^x$